TA for dynamics & inverse problem, some solutions

1 dynamics

Exercise 1. *Prove Hennion's theorem, which says the following: Let $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ be bounded on a Banach space $(\mathcal{B}, \|\cdot\|)$, and let $(\mathcal{B}', \|\cdot\|')$ be a Banach so that the inclusion $\mathcal{B} \subset \mathcal{B}'$ is compact. Assume that there exist $r_n \in \mathbb{R}$ and $R_n \in \mathbb{R}$ so that*

$$\|\mathcal{L}^n \varphi\| \leq r_n \|\varphi\| + R_n \|\varphi\|' , \qquad \forall n \geq 1 , \ \forall \varphi \in \mathcal{B} .$$

Then the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is not larger than $\liminf_{n\to\infty} (r_n)^{1/n}$.

The solution given during the TA was not complete. Beforehand, let us observe that the original proof does not go through the same argument. It is based on a characterization due to Nussbaum [Nus70], and is due to Hennion [Hen93]. Now then.

First, we claim the following. Assume that $|\lambda| > r_1$. Then, if for some bounded sequence (ϕ_n) of elements of $\mathcal{B}$, $(\mathcal{L} - \lambda)\phi_n$ converges to some $\psi \in \mathcal{B}$, then we can extract a converging subsequence of (ϕ_n).

Indeed, we can compute

$$|\lambda| \|\phi_n - \phi_m\| \leq \|(\mathcal{L} - \lambda)(\phi_n - \phi_m)\| + r_n \|\phi_n - \phi_m\| + R_n \|\phi_n - \phi_m\|'.$$

Since (ϕ_n) is bounded, we can extract to make it converge in $\mathcal{B}'$, whence we deduce that it is a Cauchy sequence in $\mathcal{B}$ also.

The next step is to prove that $\mathcal{L} - \lambda$ is then semi-Fredholm. This means that the range is closed and the kernel is finite dimensional. From the claim above, we deduce that sequences in the unit ball of the kernel have converging subsequences so Riesz's criterion implies it is finite dimensional. In particular, it is closed and we can consider the vector space $\mathcal{B}_\lambda := \mathcal{B}\setminus \ker(\mathcal{L}-\lambda)$. The image of $\mathcal{L}-\lambda$ as an operator on $\mathcal{B}$ and on $\mathcal{B}_\lambda$ is the same. The claim above also applies to $\mathcal{B}_\lambda$.

Now, take a sequence (ϕ_n) in $\mathcal{B}_\lambda$ such that $(\mathcal{L}-\lambda)\phi_n \to \psi$. If (ϕ_n) is bounded, we deduce that $\psi \in \Im(\mathcal{L}-\lambda)$. Now, if $\|\phi_n\| \to +\infty$, consider $u_n = \phi_n/\|\phi_n\|$. We see that $u_n \to 0$ which is impossible.

The index of a semi-Fredholm operator A is defined as

$$\mathrm{Ind}(A) = \dim\ker A - \mathrm{codim}\,\Im A.$$

This is a number in $\mathbb{Z}\cup\{-\infty\}$. Now we conclude using the following theorem:

Theorem. *The set of semi-Fredholm operators is open is $\mathcal{L}(\mathcal{B})$, and the index is locally constant on this set.*

We deduce that $\rho_{ess}(\mathcal{L}) \leq r_1$. Since $\rho_{ess}(\mathcal{L}^n) = \rho_{ess}(\mathcal{L})^n$, we can conclude.

The take home lesson of this is that while there are several definitions of the essential spectrum, the resulting essential spectral radius does not depend on the particular definition.

Exercise 2. *For an integer $m \geq 2$, let $f_m : \mathbb{S}^1 \to \mathbb{S}^1$ be the multiplication by m (mod 1) on the circle and let $g_m : I \to I$ be the multiplication by m (mod 1) on $I = [0,1]$.*

(*a*) *Compute the zeroes of the dynamical determinants of $F = f_m$ and $F = g_m$, weighted with $1/|F'|$,*

$$d_{F,1/|F'|}(z) = \exp -\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in \mathrm{Fix}\,F^n} \frac{1}{|(F^n)(x)' - 1|}$$

where $\mathrm{Fix}\,F^n = \{x \mid F^n(x) = x\}$.

(*b*) *For $\alpha > 0$, compute the essential spectral radius ρ_α of the transfer operator*

$$\mathcal{L}_{F,1/|F'|}\varphi(x) = \sum_{F(y)=x} \frac{\varphi(y)}{|F'(y)|}$$

acting on C^α functions, for $F = f_m$ and $F = g_m$. (Hint: for the lower bound, find an open disc in which every point is an eigenvalue of infinite *multiplicity.) Find the eigenvalues of modulus $> 1/\rho_\alpha$ of $\mathcal{L}_{F,1/|F'|}$ on C^α. Describe the corresponding eigenfunctions. Check that the dual of $\mathcal{L}_{F,1/|F'|}$ preserves Lebesgue measure. Prove that f_m and g_m each has an invariant absolutely continuous probability measure, denoted μ_{f_m} and μ_{g_m}, respectively. What can we say about the rate of decay of correlations*

$$\int \varphi \circ F^n \psi d\mu_F - \int \varphi d\mu_F \int \psi d\mu_F$$

for C^α functions φ and ψ and $F = f_m$ or $F = g_m$?

(*c*) *Compute the zeroes and poles of the dynamical zeta functions of $F = f_m$ and $F = g_m$, weighted with $1/|F'|$,*

$$\zeta_{F,1/|F'|}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, F^n} \frac{1}{|(F^n)'(x)|}\,.$$

This exercise deals with the most basic example of uniformly expanding dynamics on a compact manifold. In some sense, it corresponds to a simplified system where we have forgotten the stable manifold of the problem. As such, the definition of the transfert operator is slightly different from the invertible case. However, we expect similar phenomena to happen.

In this case, we are considering a smooth map (or a smooth by parts map in the case of g_m, but we will concentrate on the case of f_m). Its derivative being constant equal to m, it is the most simple example one can think of. As we will see, its spectrum is trivial reduced to 1. This is suiting as the dynamics is in some sense empty. It would suffice to take a generic C^2 small perturbation to create spectrum, but that is a story for another day.

Question (a). This is an explicit computation. We can see the map realized on the circle $\{x \in \mathbb{C} \mid |x| = 1\}$, as $f_m(x) = x^m$. The fix points of f_m^n are the points such that $x^{m^n} = x$, i.e, there $m^n - 1$ such points. The derivative of f_m^n being constant equal to m^n, we find directly that

$$d_{f_m,1/|f_m'|}(z) = \exp - \sum_{n \geq 1} \frac{z^n}{n} = 1 - z.$$

The determinant only has 1 zero, so we do not expect to find resonances when doing the spectral analysis of the operator.

Question (b). Having just done the exercise above, we're expecting to have to use Hennion's theorem. Let $\epsilon > 0$ be small enough (less that $0,001$), and for $0 < \alpha \leq 1$, let

$$\|\phi\|_{Lip(\alpha)} := \sup_{|x-x'|<\epsilon} \frac{|\phi(x) - \phi(x')|}{|x - x'|^\alpha}.$$

Given two points x, x' such that $|x - x'| < \epsilon$, the sets $f_m^{-1}(x)$ and $f_m^{-1}(x')$ can be decomposed into pairs of points y, y' such that $|y - y'| = |x - x'|/m$. We deduce that

$$\|\mathcal{L}\phi\|_{Lip(\alpha)} \leq \sup_{|y-y'|<\epsilon/m} \frac{|\phi(y) - \phi(y')|}{m^\alpha |y - y'|^\alpha} \leq \frac{1}{m^\alpha} \|\phi\|_{Lip(\alpha)}.$$

We deduce that

$$\|\mathcal{L}\phi\|_{C^\alpha} \leq \frac{1}{m^\alpha} \|\phi\|_{C^\alpha} + (1 - m^{-\alpha}) \|\phi\|_{L^\infty}.$$

(nothing depending on ϵ, after all). Whence we see that nothing is gained by taking iterates, and whenever $0 < \alpha \leq 1$,

$$\rho_{ess}(\mathcal{L}) \leq \frac{1}{m^\alpha}.$$

Since $\mathcal{L}(\phi)' = m^{-1}\mathcal{L}(\phi')$ we deduce that the same estimate holds for all $\alpha > 0$.

Now, for the lower bound, we try to find eigenfunctions. Working with periodic functions on the circle, we can abstractly decompose them as sums

$$\phi(x) = \sum_{k \in \mathbb{Z}} a_k x^k.$$

Now, assume $\mathcal{L}\phi = \lambda\phi$, we see that

$$\mathcal{L}\phi(x) = \sum_{k \in \mathbb{Z}} a_k x^{km},$$

hence whenever $m|k$, $a_k = \lambda a_{k/m}$, and there is no condition on a_k when $m \nmid k$. The eigenfunctions decompose as:

$$\sum_{m \nmid k} a_k \sum_{\ell \geq 0} \lambda^\ell z^{km^\ell}.$$

If these sums converge and exist as functions, we conclude that the eigenspace for λ is infinite dimensional, and $\rho_{ess}(\mathcal{L}) \geq |\lambda|$.

Whenever $|\lambda| < 1$, the sums converge, so these functions are well defined in $C^0(\mathbb{S}^1)$. However, they are not in all the Hölder spaces. Observe that

$$\|z^k\|_{Lip(\alpha)} = \sup_{|x-x'|<\epsilon} \frac{|e^{ikx} - e^{ikx'}|}{|x-x'|^\alpha} = k^\alpha \sup_{|x|<k\epsilon} \frac{|e^{ix}-1|}{|x|^\alpha} \leq Ck^\alpha.$$

In particular, for $0 < \alpha \leq 1$,

$$\|\sum_{\ell \geq 0} \lambda^\ell z^{km^\ell}\|_{Lip(\alpha)} \leq k^\alpha \sum_{\ell \geq 0} (|\lambda| m^\alpha)^\ell.$$

Hence, the whole of $\{z \mid |z| < m^{-\alpha}\}$ is in the spectrum of $\mathcal{L}$. We leave the case $\alpha > 1$ to the reader.

Actually, the discrete spectrum will not depend on the space, which implies that discrete eigenfunctions are necessarily C^∞. In particular, their Fourier series converge in C^k for all k, and tracing back the argument above, we deduce that the only discrete eigenfunction is the constant function.

The action of the dual of $\mathcal{L}$ on the Lebesgue measure μ is given by

$$\int f \mathcal{L}^* \mu = \int \mathcal{L} f \mu = \int f \mu.$$

(basic change of variable.) The invariant absolutely continuous measure is just the Lebesgue measure. Since there is no discrete spectrum apart from 1, we deduce that when $\phi, \psi \in C^\alpha$,

$$\int \phi\psi \circ f_m^n d\mu = \int \phi \int \psi + \mathcal{O}(m^{-\alpha n}).$$

Question (c).

$$\zeta(z) = \exp \sum_{n \geq 1} \frac{z^n}{n} \frac{m^n - 1}{m^n} = \frac{1 - z/m}{1 - z}.$$

Exercise 3. *Let $\mathcal{Q} : \mathcal{B} \to \mathcal{B}'$ be a bounded linear operator between Banach spaces. For $k \in \mathbb{Z}_+^*$, the k-th approximation number of $\mathcal{Q}$ is*

$$a_k(\mathcal{Q}) = \inf\{\|\mathcal{Q} - \mathcal{R}\|_{\mathcal{B} \to \mathcal{B}'} \mid \operatorname{rank}(\mathcal{R}) < k\}\,.$$

(a) *Check that* $\lim_{k\to\infty} |a_k(\mathcal{Q})| = 0$ *implies that* $\mathcal{Q}$ *is compact. (The inverse implication is not true in general!) (Hint: A finite rank operator is compact.)*

(b) *Pietsch proved (see e.g. the book "Eigenvalues and s-numbers," Cambridge University Press, 1987) that for any bounded operator* $\mathcal{Q} : \mathcal{B} \to \mathcal{B}$, *if* $a_k(\mathcal{Q}) \in \ell^1(\mathbb{Z}_+^*)$, *then* $\mathcal{Q}$ *is nuclear, and, if* $a_k(\mathcal{Q}) \in \ell^q(\mathbb{Z}_+^*)$ *for some* $q \in (0, 1]$, *then* $\mathcal{Q}$ *is q-nuclear.*

Using Pietsch's result, show that if $a_k(\mathcal{Q}) \in \ell^p(\mathbb{Z}_+^*)$ *for* $0 < p < \infty$ *then for any* $q \in (0, 1]$ *there exists* $\mathcal{N} = \mathcal{N}(p, q) < \infty$ *so that* $\mathcal{Q}^{\mathcal{N}}$ *is q-nuclear.*

Question (a). The condition $a_k(Q) \to 0$ implies that we can build a sequence of finite rank operators that converges in norm to Q. Hence Q is compact.

Question (b). We consider that when R has finite rank k,

$$(Q - R)^N = Q^N + R'$$

where R' has finite rank less or equal to kN — finite rank operators are a bilateral ideal. Hence

$$a_{kN}(Q^N) \leq a_k(Q)^N.$$

Since $(a_k(Q))_k$ is decreasing, we conclude that

$$\sum_{k\geq 1} a_k(Q^N)^q \leq \sum_{k\geq 1} a_{\lfloor k/N \rfloor}(Q)^{qN} = N \sum_{k\geq 0} a_k(Q)^{qN}.$$

Exercise 4. *Consider the matrix* $F = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ *acting on the two-torus.*

(a) *Prove that* F *is an area-preserving Anosov diffeomorphism.*

(b) *Give a complete proof of the bound on the essential spectral radius of* $\mathcal{L}\varphi(x) = (\varphi/|\det DF|) \circ F^{-1}(x)$ *acting on the Banach spaces* $\mathcal{B}^{t,s}$ *defined in the course for real numbers* $s < 0 < t$ *(or in Chapter 5.1 of the book).*

(c) *Compute the dynamical determinant*

$$d_{F^{-1},g}(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x:F^m(x)=x} \frac{\prod_{k=0}^{-(m-1)} g(F^k(x))}{|\det(\mathbb{1} - DF^{-m}(x))|}$$

and the dynamical zeta function

$$\zeta_{F^{-1},g}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{F^m(x)=x} \prod_{k=0}^{-(m-1)} g(F^k(x))$$

of F^{-1} for the weight $g = 1/|\det DF| \circ F^{-1}$. Deduce from this and the results of the course that

$$\int \varphi \circ F^n \psi dx - \int \varphi dx \int \psi dx$$

decays exponentially for C^α functions φ and ψ, with $\alpha > 0$. Give an upper bound on the rate of mixing.

Question (a). Since F is just a matrix, it is equal to its differential. It preserves the volume because its determinant is 1. For it to be Anosov, it suffices to check that the eigenvalues have modulus not equal to 1. Then, the eigenspaces will be the stable/unstable directions. Since the trace is 3, the eigenvalues cannot have modulus 1. Actually, they are

$$\lambda_\pm = \frac{3 \pm \sqrt{5}}{2}.$$

Question (b). The space $\mathcal{B}^{t,s}$ is just $W^{t,s,p=2}$. Since here the dynamics is conservative, the formula is quite simpler, since the weight $g = 1/|\det DF| = 1$. The theorem states that

$$\rho_{ess}(\mathcal{L}) \leq \lim_{m\to\infty} \left(\max(\lambda_-^{mt}, \lambda_+^{ms}\right)^{1/m} = \lambda_-^{\max(t,-s)}.$$

In our case, since the manifold is just the simple torus $\mathbb{T}^2$, and since the stable/unstable manifolds are constant, we can build the spaces $\mathcal{B}^{t,s}$ more easily. We pick a $[0,1]$-valued function ψ_+ on the circle, so that $\psi_+(\xi/|\xi|) = 1$ around $\{\xi F^{-1} = \lambda_-\xi\}$, and 0 around $\{\xi F^{-1} = \lambda_+\xi\}$. Let $\psi_- = 1 - \psi_+$. For $\varphi \in C^\infty(\mathbb{T}^2)$, we define

$$\|\varphi\|^2_{\mathcal{B}^{t,s}} := |\widehat{\varphi}(0)|^2 + \sum_{k\neq 0} \left[(k^2)^{t/2}\psi_+(k/|k|) + (k^2)^{s/2}\psi_-(k/|k|)\right] |\widehat{\varphi}(k)|^2.$$

Then

$$\begin{aligned}\|\mathcal{L}^n\varphi\|_{\mathcal{B}^{t,s}} &= |\widehat{\varphi}(0)|^2 \\ &+ \sum_{k\neq 0} \left[\|kF^{-n}\|^t\psi_+(kF^{-n}/|kF^{-n}|) + \|kF^{-n}\|^s\psi_-(kF^{-n}/|kF^{-n}|)\right] |\widehat{\varphi}(k)|^2\end{aligned}$$

Writing $k = \alpha_+ k_+ + \alpha_- k_-$, where $k_\pm$ are normalized orthogonal eigenvectors for F^{-1} — it is symmetric — ψ_+ writes as a smooth function $\tilde{\psi}_+$ of α_-/α_+. We can rewrite the coefficients of $|\widehat{\varphi}(k)|^2$ — $k \neq 0$ — as

$$\psi_+\left(\frac{\alpha_-}{\alpha_+}\right)(\alpha_+^2 + \alpha_-^2)^{t/2} + \psi_-\left(\frac{\alpha_-}{\alpha_+}\right)(\alpha_+^2 + \alpha_-^2)^{s/2}$$
$$\psi_+\left(\frac{\lambda_-^n \alpha_-}{\lambda_+^n \alpha_+}\right)(\lambda_+^{2n}\alpha_+^2 + \lambda_-^{2n}\alpha_-^2)^{t/2} + \psi_-\left(\frac{\lambda_-^n \alpha_-}{\lambda_+^n \alpha_+}\right)(\lambda_+^{2n}\alpha_+^2 + \lambda_-^{2n}\alpha_-^2)^{s/2}$$

Consider the first term in the second line:

$$\lambda_-^{nt}|\alpha_-|^t \left(1 + \left[\frac{\lambda_+^n}{\lambda_-^n}\frac{\alpha_+}{\alpha_-}\right]^2\right)^{t/2} \psi_+\left(\frac{\lambda_-^n \alpha_-}{\lambda_+^n \alpha_+}\right).$$

Since ψ_+ vanishes at 0, this can only be non zero when $\delta\alpha_+ < \alpha_-$ for some $\delta > 0$. We deduce that the first term of the second line is lesser or equal to the first term of the *first* line times $C\lambda_-^{nt}$ for some $C > 0$.

As to the second term of the second line, when separate two cases. First, assume that $\psi_-(\alpha_-/\alpha_+) > \epsilon$. Then $\psi_-(\lambda_-^n\alpha_-/(\lambda_+^n\alpha_+)) = 1$ — assuming n is large, independently of ϵ — and $\delta\alpha_- < \alpha_+$ for some $\delta > 0$. As a consequence, the second term of the second line is lesser or equal to the second term of the first line times $C\lambda_+^s$ for some $C > 0$.

It remains to deal with the case $\psi_-(\alpha_-/\alpha_+) \leq \epsilon$, i.e $\delta\alpha_- > \alpha_+$. Then, $\psi_+(\alpha_-/\alpha_+) \neq 0$, and

$$\psi_-\left(\frac{\lambda_-^n \alpha_-}{\lambda_+^n \alpha_+}\right)(\lambda_+^{2n}\alpha_+^2 + \lambda_-^{2n}\alpha_-^2)^{s/2} \leq C\lambda_+^{ns}\psi_+\left(\frac{\alpha_-}{\alpha_+}\right)(\alpha_+^2 + \alpha_-^2)^{t/4}$$

In particular, we conclude that for any $s' < s$, there are constants C, C_n such that

$$\|\mathcal{L}^n\varphi\|_{\mathcal{B}^{t,s}} \leq C\max(\lambda_-^{nt}, \lambda_+^{ns})\|\varphi\|_{\mathcal{B}^{t,s}} + C_n\|\varphi\|_{\mathcal{B}^{t/2,s'}}$$

We conclude the proof using Hennion's theorem.

Question (c). The only thing that we have to do is to compute the number of fixed points of F^m. This is the number of points $z \in \mathcal{C} := \{0 \leq x < 1,\ 0 \leq y < 1\}$ such that $(F^m - 1)z \in \mathbb{Z}^2$. In other words, the number of points $z' \in \mathbb{Z}^2$ so that $(F^m - 1)^{-1}z' \in \mathcal{C}$. This number is $|\det(F^m - 1)|$. One way to prove it is to consider

$$\begin{aligned}
\#\{z \in \mathbb{Z}^2 \mid (F^m - 1)^{-1} z \in \mathcal{C}\} &= \frac{\#\{z \in \mathbb{Z}^2 \mid (F^m - 1)^{-1} z \in n\mathcal{C}\}}{n^2} \\
&\underset{n\to+\infty}{\sim} \operatorname{vol}((F^m - 1)\mathcal{C}).
\end{aligned}$$

To finish the computation, observe that we also have $|\det 1 - F^m| = |\det 1 - F^{-m}|$. Hence

$$d_{F^{-1},1}(z) = 1 - z.$$

and

$$\begin{aligned}
\zeta_{F^{-1},1}(z) &= \exp \sum_{m\geq 1} \frac{z^m}{m}(\lambda_+^m + \lambda_-^m - 2) \\
&= \frac{(1-z)^2}{(\lambda_+ - z)(\lambda_- - z)}.
\end{aligned}$$

Since there is only 1 resonance, we deduce that when $\phi, \psi \in \mathcal{B}^{t,-t}$,

$$Cov(\phi, \psi, n) = \mathcal{O}(\lambda_-^{nt}).$$

It remains to determine which space $\mathcal{B}^{t,s}$ contains C^α. Since we are on a compact manifold, certainly, $C^\alpha \hookrightarrow H^\alpha$ with α integer, and by interpolation, we deduce the same holds when $\alpha \in \mathbb{R}$. Since $H^s \subset \mathcal{B}^{s,-s}$, the rate of decay is at least $\lambda_-^{\alpha n}$.

2 Inverse problems

Exercise 5 (Computing symbols I)**.** *Consider the case of* $\mathbb{R}^2$*, recall*

$$I_0 f(\theta, s) = \int_{x\cdot\theta = s} f.$$

We take the adjoint with respect to the usual measure $d\theta ds$ *on* $\mathbb{S}^1 \times \mathbb{R}$*. Prove*

$$\frac{1}{2}\Delta^{1/2} I_0^* I_0 = \mathbb{1}.$$

(think of Fourier transforms).

This is a big change of variables. First, we compute the kernel of $I_0^* I_0$, and then we check that this is indeed the kernel of $2\Delta^{-1/2}$. Take $f \in C_c^\infty(\mathbb{R}^2)$, and consider

$$\begin{aligned}
\|I_0 f\|^2 &= \int d\theta ds \int f(s\theta + i\lambda\theta)\overline{f(s\theta + i\lambda'\theta)} d\lambda d\lambda' \\
&= \int d\theta \int dz f(z) \int d\lambda'' \overline{f(z + i\lambda'\theta)}.
\end{aligned}$$

Hence $I_0^* I_0 f(z)$ is

$$2 \int f(z+u)\frac{du}{|u|}.$$

Since this a convolution kernel, $I_0^* I_0$ is a Fourier multiplier, and the multiplier is the distribution

$$K(\xi) = \mathcal{F}(1/|z|)(\xi).$$

We see that $K(\xi)$ is invariant under rotations and that $|\xi|K(\xi) = C$ by a change of variable. In particular, $K(\xi) = C/|\xi| + A$, where A has to be a distribution supported at 0, and invariant by rotations. The only possible case is $A = 0$. To find the constant, we consider that

$$\mathcal{F}^2(1/|z|) = C^2/|z|,$$

so $C = 1$ ($I_0^* I_0$ is positive).

For the rest of the exercises, we take some Riemannian surface with boundary (M, g).

Exercise 6. *Verify that strict infinitesimal convexity implies the convexity of the interior of M. In other words, assuming that the boundary is strictly convex — $\mathrm{II} > 0$ — show that there are no geodesics of M tangent to its boundary, and not reduced to a point.*

The proof is not very hard, but some concepts have to be clearly defined. Let N be an immersed submanifold in M. I.e, N is a manifold, and we have a submersion $i : N \to M$. Take a vector field X on N, a connection ∇ on M, and Y a section of $TM_{|N}$. We can see N locally as a submanifold and extend X, Y to a neighbourhood. Then we set

$$i^* \nabla_X Y = \nabla_X Y_{|N}.$$

This is a well defined connection on $TM_{|N}$ — it suffices to take charts, and check that the value of $i^*\nabla_X Y(n)$ only depends on the value of X at n and the 1-jet of Y in the direction tangent to N. For details: locally, $\nabla_X Y$ writes as $dY(X) + A(Y, X)$ where A is a linear expression of the 0-jet of X, Y.

Now, given X, Y two vector fields on N, we can see TN as a subbundle of $TM_{|N}$, so the following makes sense:

$$II(X, Y) := (i^*\nabla_X Y)^\perp.$$

We have to check that if $Z \perp TN$, $\langle II(X, Y), Z\rangle$, so that II is a bilinear form on TN. In local charts, this is straightforward. When N is an hypersurface, we also denote by II the scalar product of II with a normal vector — a trivialization of the normal bundle.

In particular, if $\gamma(t)$ is a non-stationary curve on M, seeing the map γ as a submersion, γ is a geodesic if and only if

$$\gamma^*\nabla_{\frac{d}{dt}}\dot{\gamma} = 0.$$

This is colloquially written as $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

Now, consider a geodesic $\gamma(t)$ living in a small neighbourhood of ∂M. Let $r(x) = d(x, \partial M)$, and $r(t) = r(\gamma(t))$. This is locally defined. Then

$$\begin{aligned}
\frac{d^2}{dt^2}r(t) &= \frac{d}{dt}\langle \nabla r(\gamma(t)), \dot{\gamma}(t)\rangle \\
&= \langle \gamma^*\nabla_{\frac{d}{dt}}\nabla r, \dot{\gamma}(t)\rangle + \langle \nabla r, \gamma^*\nabla_{\frac{d}{dt}}\dot{\gamma}\rangle \\
&= \langle \nabla_{\dot{\gamma}}\nabla r, \dot{\gamma}\rangle
\end{aligned}$$

Now, consider the bilinear form

$$A(X, Y) := \langle \nabla_X \nabla r, Y\rangle.$$

Since the Levi-Civita connection is torsion-free, this is symmetrical. Additionnally,

$$A(X, \nabla r) = 0.$$

When $X_{|\partial M}, Y_{|\partial M}$ are tangent to ∂M, writing

$$A(X, Y) = X\langle \nabla r, Y\rangle - \langle \nabla r, \nabla_X Y\rangle,$$

We see that

$$A(X, Y) = -\langle II(X_{|\partial M}, Y_{|\partial M}), \nu\rangle + \mathcal{O}(r),$$

where ν is the inward pointing normal.

Now, we conclude. If $\dot\gamma$ is close to ∇r, then γ is not tangent to the boundary. On the other hand, if $\dot\gamma$ is a open cone around $T\partial M$, $\ddot r < 0$, since $II > 0$. In that case, $r(t)$ is locally a strictly concave function of t, and γ cannot be tangent to the boundary.

Exercise 7 (Integrating the flow)**.** *Recall* $\Pi = I^*I$, *and*

$$R_\pm f = \pm \int_0^{\pm\infty} f \circ \varphi_t dt.$$

Assume that $f \in C^\infty(SM)$.

(a) *Check that*

$$\Pi f = (R_+ - R_-)f.$$

(b) *Deduce that* $f \in \ker I$ *if and only if there exists* $u \in C^\infty(SM \setminus \partial_0 SM) \cap C^0(SM)$, *vanishing at the boundary with* $-Xu = f$.

Here, we are assuming that the manifold is *non-trapping.*

Question (a). The measure on the boundary is $(v \cdot \nu)dv$. Both operators $R_\pm$ are well defined, and we have

$$(R_+ - R_-)f = \int_{-\infty}^{+\infty} f \circ \varphi_t dt.$$

On the other hand, we consider:

$$\|If\|^2 = \int_{\partial^- SM} v \cdot \nu dv \int dt dt' f(\varphi_t(v))\overline{f(\varphi_{t'}(v))}.$$

Consider $A : t, v \mapsto w = \varphi_t(v)$. This is a legal change of variables. Since φ_t preserves the volume of SM, we find that $\mathrm{Jac}(\varphi_T \circ A) = \mathrm{Jac}\, A$. In particular, the jacobian of A does not depend on t, and we can compute it at $t = 0$, i.e on the boundary. There, for small times,

$$A(t; (x, 0, v)) = (x + t(v - v \cdot \nu)\nu, t(v \cdot \nu), v).$$

In particular, $\mathrm{Jac}(A) = v \cdot \nu$, and $A_*(v \cdot \nu)dtdv = dw$. Thus, we can rewrite

$$\|If\|^2 = \int dw f(w) \overline{\int f(\varphi_{t'}(w))dt'}.$$

Question (b). First, let us assume such a u exists. Then $u(v)-u(\varphi_t(v)) = \int_0^t f(\varphi_s(v))ds$. In particular, if $v_\pm \in \partial^\pm SM$ are the endpoints of some geodesic,

$$If(v_-) = -u(v_+) + u(v_-).$$

Since u vanishes at the boundary, $If = 0$.

Now, assume that f is C^∞, and $f \in \ker I$. Then $\Pi f = (R_+ - R_-)f = 0$. Set

$$u = R_+ f = R_- f.$$

Since geodesic intersect the boundary transversally, we find that in all points of the interior, u is C^∞, and $Xu = -f$. Since it is equal to $R_+ f$, it vanishes at $\partial^+SM \cup \partial^0 SM$, and likewise at $\partial^- SM$ using $R_- f$. Actually, taking charts close to the boundary, we see that it is actually smooth up to the boundary, except at the glancing set $\partial^0 SM$.

Something can probably be said about the behaviour at the glancing set, but that is beyond the scope of the exercise.

Exercise 8 (Computing some wavefront sets)**.** *Recall Jared's talk and*

(*a*) *Prove that if* $u \in \mathcal{D}'(X \times Y)$ *and* $\pi : X \times Y \to Y$ *is the right projection,*

$$WF(\pi_* u) = d\pi(WF(u) \cap N^* X).$$

(*b*) *If* ψ_t *is a smooth flow, compute its wavefront set.*

(*c*) *Deduce the wavefront set of* Π.

Question (a). Obviously, one should prove that the LHS is included in the RHS. The equality is not true in general. We can cut X, Y into small pieces, where we have charts to $\mathbb{R}^n$. Taking a decomposition of unity, we reduce to the case of $u \in \mathcal{D}'(\mathbb{R}^k \times \mathbb{R}^n)$, $\psi \in C_c^\infty(\mathbb{R}^k)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$, and we want to compute the wavefront set of

$$f \in C^\infty(\mathbb{R}^n) \mapsto \int u(x,y)\psi(x)\phi(y)f(y)dxdy.$$

We can directly go to Fourier transforms, because localizing the distribution will just give another of the same type. Hence we apply this distribution to $f = e^{-i\lambda y}$, and obtain

$$\mathcal{F}(u\psi\phi)(0,\lambda).$$

In particular, this is rapidly decreasing if

$$WF(u) \cap \{(x, y, 0, \lambda) \mid x \in \operatorname{supp} \psi,\ y \in \operatorname{supp} \phi\} = \emptyset.$$

This is exactly the statement we set out to prove. The notation N^*X is the conormal bundle to X, i.e linear forms that vanish on TX. More generally, the same result (with the same proof) holds for general fibrations.

Question (b). We consider an open set $U \subset \mathbb{R}^n$, and a flow φ_t on U with non-vanising vector field X — not necessarily complete. Let $\Omega \subset \mathbb{R} \times U$ be its domain of definition. Ω is open. We want to find the wavefront set of

$$B : \psi \in C_c^\infty(\Omega \times U) \mapsto \int \psi(t, x, x')\delta(x' - \varphi_t(x))dx'dxdt.$$

We take ϕ, ψ compactly supported functions in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, and compute

$$\int e^{-i(\lambda(t-t')+\mu(x-x')+\mu'(y-y'))/h}\phi(t, x, y)\psi(\lambda, \mu, \mu')\delta(y - \varphi_t(x))dtdxdyd\lambda d\mu d\mu'.$$

This is

$$\int e^{-i(\lambda(t-t')+\mu(x-x')+\mu'(\varphi_t(x)-y'))/h}\phi(t, x, \varphi_t(x))\psi(\lambda, \mu, \mu')dtdxd\lambda d\mu d\mu'.$$

We can use stationary phase in the variables t, x, λ, μ, at the point $t = t', x = x', \lambda = -\mu' X(\varphi_t(x)), \mu = -\mu' d_x\varphi_t$. We find an expansion in the form

$$\int \frac{h^{n+1}e^{i\Phi(\mu')/h}}{C(\mu')}\phi(t', x', \varphi_t(x'))\psi(-\mu' X(\varphi_t(x)), -\mu' d_x\varphi_t, \mu')d\mu'$$

As a consequence,

$$WF(B) = \{(t, x, x'; \lambda, \mu, \mu') \mid x' = \varphi_t(x),\ \lambda = -\mu' X(x'),\ \mu = -\mu' d_x\varphi_t\}.$$

Question (c). If π is the projection $\mathbb{R} \times SM \times SM$, we see that the kernel of Π is $\pi_* B$. Hence

$$\begin{aligned} WF(\Pi) &\subset d\pi(WF(B) \cap \{\lambda = 0\}), \\ &\subset \{(v, v'; \mu, \mu') \mid \exists t,\ v' = \varphi_t,\ \mu'(X(v')) = 0,\ \mu = -\mu' d_v\varphi_t\}. \end{aligned}$$

Exercise 9 (Computing symbols II)**.** *If $\pi_0 : TM \to M$ is the projection, $I_0 = I\pi_0^*$ and $\Pi_0 = I_0^* I_0$.*

(a) *Find the wavefront set of* Π_0.

(b) *When there are no conjugate points, check that* Π_0 *is a pseudor, and compute the principal symbol.(cut the operator into a smoothing part and another supported close to the diagonal).*

(c) *More generally, if* χ *is a cutoff around a transient trajectory, show* $I_0^* \chi^2 I_0$ *is a pseudor.*

Question (a). Again, we use the projection lemma: if $\pi' : SM \times SM \to M \times M$ is the projection in both variables, the kernel of Π_0 is $\pi'_* K_\Pi$. If $(x, \xi) \in T^*M$ and $v \in S_xM$, we denote by $\mu(\xi) \in T^*_{(x,v)}SM$ the only point in the conormal to S_xM that projects down to ξ. In particular,

$$\begin{aligned} WF(\Pi_0) \subset \{(x, x'; \xi, \xi') \mid \exists t, v, v', \\ \varphi_t(x, v) = (x', v'),\ \mu(\xi')(X(x')) = 0,\ \mu(\xi) = -\mu(\xi') d_{(x,v)} \varphi_t\}. \end{aligned}$$

Take a point $(x, x'; \xi, \xi')$ in the wavefront set, and assume that the corresponding t is not 0. Then we can find μ' so that $\mu'(\partial_v) = 0$ — it is conormal to $S_{x'}M$ — and also $\mu' d_x \varphi_t \partial_v = 0$. Consider the Jacobi field along the geodesic between x and x', $J(s) = d\pi d_x \varphi_s \partial_v$. It has to vanish at $s = 0$, and the condition above shows that it also vanishes at $s = t$. In particular, x and x' are conjugate points.

Now, if $t = 0$, then $x = x'$, and the condition on ξ, ξ' is just that $\xi = \xi'$. We conclude that

$$\begin{aligned} WF(\Pi_0) \subset \{(x, x; \xi, \xi) \mid (x, \xi) \in T^*M\} \cup \{(x, x', \xi, \xi') \mid \\ x, x' \text{ are conjugate points along } \varphi_t(x, v), \xi \cdot v = 0, \xi' = d\pi d_{x,v} \varphi_t d\pi^{-1} \xi\}. \end{aligned}$$

Question (b). When there are no conjugate points, The wavefront set of Π_0 is restricted to the diagonal. Recall that

$$\Pi_0 f(x) = \int f(\varphi_t(x, v)) dt dv.$$

Take $\chi \in C_c^\infty(\mathbb{R})$, equal to 1 in a neighbourhood of 0, and consider the decomposition:

$$\Pi_0 = \underbrace{\int \chi(t) f(\varphi_t(x, v)) dt dv}_{\Pi_0^1} + \underbrace{\int (1 - \chi(t)) f(\varphi_t(x, v)) dt dv}_{\Pi_0^2}.$$

If we are working at a point x not on the boundary, if the support of χ is small enough, we can study Π_0^1 forgetting the boundary. As we have seen from the arguments above, we also already know that Π_0^2 is a smoothing operator.

We want to show that Π_0 acts as a pseudo-differential operator on $C_c^\infty(S\mathring{M})$, so it suffices to consider Π_0^1, and we can forget that there is a boundary.

In this small neighbourhood of x, $\exp_x^{-1}$ is a diffeomorphism, so we take as a chart, and

$$\Pi_0^1 f(x) = 2\int \chi(t) f(\exp_x(tv)) dt dv = 2\int \frac{\chi(d(x,x'))}{d(x,x')} f(x') \frac{dx'}{\Theta(x,x')},$$

where $\Theta(x,x') = \det T_{\exp_x^{-1}(x')} \exp_x$. We have $\Theta(x,x) = 1$, and Θ is C^∞ in a neighbourhood of the diagonal.

To check that Π_0^1 is a pseudo-differential operator, we pick a chart x in which the riemannian measure is mapped to dx. If the kernel is denoted by K, we compute

$$a(x,\xi) := \int K\left(x + \frac{u}{2}, x - \frac{u}{2}\right) e^{-iu\xi} du.$$

Since $d(x+u/2, x-u/2) = |u|_x g(x,u)$ where $g(x,u)$ is a smooth non vanishing function, equal to 1 at $u = 0$, this is of the form

$$a(x,\xi) = 2\int \frac{e^{-iu\cdot\xi} F(x,u)}{|u|_x} du,$$

Where $F(x,u)$ is smooth compactly supported in the u variable, and is equal to 1 at $u = 0$. We deduce that $a(x,\xi)$ is smooth. To conclude, we need to obtain symbolic estimates, and an asymptotics as $\xi \to \infty$.

Rephrasing the formula, we get

$$a(x,\xi) = 2\widehat{F_x} * \frac{1}{|\eta|_x}(\xi)$$

Since $\widehat{F_x}$ is smooth family (depending on x) of Schwartz function, and since $1/|\eta|_x$ is locally L^1, and is a symbol outside of a neighbourhood of 0, we get directly that a is a symbol, and also that

$$a(x,\xi) = \frac{1}{|\xi|_x} \underbrace{\int \widehat{F_x}(\eta) d\eta}_{=F(x,0)=1} + \mathcal{O}(|\xi|^{-2})$$

Question (c). Now we have removed the non-trapping assumption, put we have a small cutoff χ, localizing in $\partial^- SM$ around a non-trapped trajectory. We also assume that along that geodesic there are no conjugate points. Since this is an open condition, shrinking the support of χ, we can assume that there are no conjugate points along any geodesic starting in the support of χ.

If for a point (x, v) not trapped in the past, let (x_-, v_-) be the corresponding point in $\partial^- SM$. We define $\chi'(x, v)$ as $\chi(x_-, v_-)$ where (x_-, v_-) is the point of intersection of the geodesic through (x, v) with $\partial^- SM$, if it exists. This is a smooth function on SM. we have

$$I_0^* \chi I_0 f(x) = \int f(\varphi_t(x, v))\chi'(x, v) dt dv.$$

Doing the same decomposition as before, and again taking charts, we are led to considering the operator

$$Kf(x) = \int f(x')\chi'' \left(x, \frac{x - x'}{|x - x'|_x} \right) \frac{F(x, x')dx'}{|x - x'|_x},$$

where χ'' is smooth, F also, and $F(x, x) = 1$. Consider

$$\int e^{iu\xi} K(x, x + u) du.$$

This is a well behaved symbol if and only if

$$\int e^{iu\xi} \frac{g(u/|u|)}{|u|} du$$

behaves as a symbol at $\xi \to \infty$, for any g smooth. But this takes the form

$$\tilde{g}(\xi/|\xi|)|\xi|^{-1}.$$

This ends the proof.

Exercise 10 (Applications in the non-trapping case)**.** *When there are neither conjugate points, nor trapped geodesics,*

(a) *Show that functions in* $\ker I_0 \cap L^1(M)$ *are smooth up to the boundary. (hint: imagine that you know that they have to vanish to all order at the boundary)* (*b*) *Assuming that* I_0 *is injective, show that* I_0^* *is surjective from* $H^{s-1/2}(\partial SM)$ *to* $H^s(M)$ *(provided* $s > 1/2$*).*

Question (a). Consider $f \in \ker I_0$. Then also $f \in \ker \Pi_0$, and this makes sense because $f \in L^1$. Now, since Π_0 is a pseudo-differential operator with an elliptic principal symbol, we expect that this will give strong results on f. However, we have to deal with the fact that there is a boundary.

To overcome this problem, we will extend the *surface*. For simplicity, let us assume that there is only one connected component of the boundary. Close to the boundary, the manifold M is diffeomorphic to some $[0, \epsilon) \times \mathbb{S}^1$, so that we can map M to a slightly larger surface M_ϵ, taking the form $[-\epsilon, \epsilon) \times \mathbb{S}^1$ near the boundary. We can continue the metric g to M_ϵ, so that the boundary of M_ϵ still is strictly convex. We add an ϵ exponent to objects defined with respect to M_ϵ.

In that case, continuing f by 0 to obtain f^ϵ, we find that $\Pi_0^\epsilon f^\epsilon = 0$. If we were able to show that f^ϵ is smooth on the interior of M_ϵ, using techniques that apply to f also, we can conclude that f is a actually smooth *up to the boundary*. So we forget the ϵ's and try to show that f is smooth in the interior of M. Let $\delta > 0$, and let us concentrate on $\{x \in M \mid d(x, \partial M) > \delta\}$. Once again, we use the decomposition

$$\Pi_0 = \Pi_0^1 + \Pi_0^2.$$

Here Π_0^1 is supported in a $\delta/10$ neighbourhood of the diagonal, and Π_0^2 is smoothing — in the sense that $\Pi_0^2 : C^{-\infty}(M) \to C^\infty(\mathring{M})$. Since Π_0^1 has a pseudo-differential behaviour, we can pick a quantization Op supported in a $\delta/10$ neighbourhood of the diagonal, and find a symbol p^1 so that at least at $\delta/5$ from the boundary,

$$\Pi_0^1 \operatorname{Op}(p) = \mathbb{1} + \text{smoothing}.$$

We have to assume that $p \in S^1$, and the smoothing remainder is properly supported — in a $\delta/5$ neighbourhood of the diagonal. In particular, since $f \in \ker I_0$, we find

$$f + g = 0,$$

where $g \in C^\infty(\mathring{M})$.

Question (b). Since we are dealing with elliptic pseudo-differential operators, and we have an injectivity hypothesis, it is not a stretch to imagine

[1]since the principal symbol of Π_0^1 is $|\xi|^{-1}$, it is elliptic

that the answer to this question will rely on proving that an operator somewhere is Fredholm of index 0. What we would like is to show that Π_0 itself is Fredholm of index 0, deduce that it is surjective, and conclude for I_0^*. However, because of the boundary, we do not have the necessary tools. Instead of trying to make a refined study at the boundary, we change the point of view.

Again, let M_ϵ be an extension of the surface. Also let M_c be a manifold containing M_ϵ, but without boundary — if you will, we are glueing a half-sphere to each boundary component. We can build a pseudo-differential operator $\widetilde{\Pi}_0$ that coincides with Π_0 on functions supported in M. It suffices to pick a smooth compactly supported function $\chi \in C^\infty(\mathring{M}_\epsilon)$, equal to 1 in M, a Weyl quantization Op^w on M_c, and let

$$\widetilde{\Pi}_0 = \chi \Pi_0^\epsilon \chi + (1-\chi)\,\mathrm{Op}^w((1+|\xi|^2)^{-1/4})^2(1-\chi).$$

We already know that $\widetilde{\Pi}_0$ is a elliptic pseudo-differential operator of order -1 acting on a compact manifold. The parametrix construction

$$\widetilde{\Pi}_0\,\mathrm{Op}(\sigma) = \mathbb{1} + K,$$

with K smoothing and σ some symbol with principal order $|\xi|$, shows that $\widetilde{\Pi}_0$ is Fredholm of index 0 as an operator $H^s(M_c) \to H^{s+1}(M_c)$ for all $s \in \mathbb{R}$ — smoothing operators on *closed* manifolds are compact.

Now, we want to show that $\widetilde{\Pi}_0$ is injective. Take f in the kernel and consider

$$0 = \langle \widetilde{\Pi}_0 f, f\rangle = \|I_0^\epsilon \chi f\|^2 + \|\,\mathrm{Op}^w((1+|\xi|^2)^{-1/4})(1-\chi)f\|^2.$$

We deduce that $I_0^\epsilon \chi f = \mathrm{Op}^w((1+|\xi|^2)^{-1/4})(1-\chi)f = 0$.

Actually, adding a $C(-\Delta+1)^{-1/2}$ term to $\mathrm{Op}^w((1+|\xi|^2)^{-1/4})$, we are not changing the principal symbol, but when C is sufficiently large, we ensure that the resulting operator is injective — using the sharp Gårding inequality. In particular, we can assume that $(1-\chi)f = 0$.

Looking closer, we could also have assumed that $1-\chi$ is supported exactly in $M_c \setminus M$. As a consequence, f has to be supported in M. Then, $I_0^\epsilon \chi f = 0$ implies that $I_0 f_{|M} = 0$, and $f = 0$.

Since $\widetilde{\Pi}_0$ is injective, it is surjective, so given $f \in H^s(M)$, we can extend it to M_c as $\tilde{f}$, supported in M_ϵ, and find $h \in H^{s-1}$ so that $\widetilde{\Pi}_0 h = f$. Consider $(I^\epsilon)^* I_0^\epsilon \chi h = g$. This is a H^s function on SM_ϵ. We can restrict it to ∂SM, to a $g' \in H^{s-1/2}(\partial SM)$, provided $s > 1/2$ — indeed, ∂SM is a nice embedded surface in the three-dimensional manifold SM_ϵ. We claim that $I_0^* g' = f$. This just means that

$$I^*(g_{|\partial SM}) = (I^\epsilon)^*(g_{|\partial SM_\epsilon})_{|SM}.$$

This is obviously true for smooth functions, so it also has to be true for H^s.

Exercise 11. *Without assumption on conjugate points, show that*

$$I_0^* I_0 : H^{-1/2}_{comp}(M) \to H^{1/2}_{loc}(M).$$

Consider $u, v \in H^{-1/2}_{comp}(M)$.

$$|\langle I_0^* I_0 u, v\rangle|^2 = |\langle I_0 u, I_0 v\rangle|^2 \leq \|I_0 u\|^2 \|I_0 v\|^2.$$

In particular, to prove the desired result, it suffices to prove that $I_0^* I_0$ is bounded as a quadratic form on $H^{-1/2}_{comp}(M)$, i.e bound

$$q(u) := \langle I_0^* I_0 u, u\rangle.$$

Now, assume that I_0 decomposes as $A_1 + A_2 + \cdots + A_n$. Then

$$q(u) \leq n \sum q_i(u),$$

where $q_i(u) = \langle A_i^* A_i u, u\rangle$. This trick enable us to get rid of the non-diagonal terms $\langle A_i u, A_j u\rangle$ that we would not know how to deal with.

Now, we choose the decomposition, trying to avoid conjugate points. Since the manifold is non-trapping, we know that given a point $v \in S^*M$, we can find a small open set U_v around v, and $t_0 > 0$ so that $\varphi_t(U_v) \cap U_v = \emptyset$ when $|t| > t_0$. We cover S^*M by a finite number of such open sets, and pick a corresponding decomposition of unity $1 = \sum \chi_i$. Seeing the χ_i's as order 0 symbols, We let

$$A_i = I_0 \operatorname{Op}^w(\chi_i).$$

We have taken a partition of unity in the cotangent bundle because there might be a point x in M that is self-conjugate along a geodesic. Now, from the wavefront-set result we have on Π, we deduce that $A_i^* A_i$ is a pseudo-differential operator of order -1 — and principal symbol $|\xi|^{-1}\chi^2$. In particular, q_i is bounded on $H^{-1/2}_{comp}(M)$.

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