TA for dynamics & inverse problem

1 dynamics

Exercise 1. *Prove Hennion's theorem, which says the following: Let $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ be bounded on a Banach space $(\mathcal{B}, \|\cdot\|)$, and let $(\mathcal{B}', \|\cdot\|')$ be a Banach so that the inclusion $\mathcal{B} \subset \mathcal{B}'$ is compact. Assume that there exist $r_n \in \mathbb{R}$ and $R_n \in \mathbb{R}$ so that*

$$\|\mathcal{L}^n \varphi\| \leq r_n \|\varphi\| + R_n \|\varphi\|', \qquad \forall n \geq 1, \forall \varphi \in \mathcal{B}.$$

Then the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is not larger than $\liminf (r_n)^{1/n}$.

Exercise 2. *For an integer $m \geq 2$, let $f_m : \mathbb{S}^1 \to \mathbb{S}^1$ be the multiplication by m (mod 1) on the circle and let $g_m : I \to I$ be the multiplication by m (mod 1) on $I = [0, 1]$.*

(*a*) *Compute the zeroes of the dynamical determinants of $F = f_m$ and $F = g_m$, weighted with $1/|F'|$,*

$$d_{F,1/|F'|}(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, F^n} \frac{1}{|(F^n)(x)' - 1|}$$

where $\mathrm{Fix}\, F^n = \{x \mid F^n(x) = x\}$.

(*b*) *For $\alpha > 0$, compute the essential spectral radius ρ_α of the transfer operator*

$$\mathcal{L}_{F,1/|F'|}\varphi(x) = \sum_{F(y)=x} \frac{\varphi(y)}{|F'(y)|}$$

acting on C^α functions, for $F = f_m$ and $F = g_m$. (Hint: for the lower bound, find an open disc in which every point is an eigenvalue of infinite *multiplicity.) Find the eigenvalues of modulus* $> 1/\rho_\alpha$ *of* $\mathcal{L}_{F,1/|F'|}$ *on* C^α*. Describe the corresponding eigenfunctions. Check that the dual of* $\mathcal{L}_{F,1/|F'|}$ *preserves Lebesgue measure. Prove that* f_m *and* g_m *each has an invariant absolutely continuous probability measure, denoted* μ_{f_m} *and* μ_{g_m}*, respectively. What can we say about the rate of decay of correlations*

$$\int \varphi \circ F^n \psi d\mu_F - \int \varphi d\mu_F \int \psi d\mu_F$$

for C^α *functions* φ *and* ψ *and* $F = f_m$ *or* $F = g_m$*?*

(*c*) *Compute the zeroes and poles of the dynamical zeta functions of* $F = f_m$ *and* $F = g_m$*, weighted with* $1/|F'|$,

$$\zeta_{F,1/|F'|}(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in \mathrm{Fix}\, F^n} \frac{1}{|(F^n)'(x)|}\,.$$

Exercise 3. *Let* $\mathcal{Q} : \mathcal{B} \to \mathcal{B}'$ *be a bounded linear operator between Banach spaces. For* $k \in \mathbb{Z}_+^*$*, the* k*-th approximation number of* $\mathcal{Q}$ *is*

$$a_k(\mathcal{Q}) = \inf\{\|\mathcal{Q} - \mathcal{R}\|_{\mathcal{B}\to\mathcal{B}'} \mid \mathrm{rank}(\mathcal{R}) < k\}\,.$$

(*a*) *Check that* $\lim_{k\to\infty} |a_k(\mathcal{Q})| = 0$ *implies that* $\mathcal{Q}$ *is compact. (The inverse implication is not true in general!) (Hint: A finite rank operator is compact.)*

(*b*) *Pietsch proved (see e.g. the book "Eigenvalues and s-numbers," Cambridge University Press, 1987) that for any bounded operator* $\mathcal{Q} : \mathcal{B} \to \mathcal{B}$*, if* $a_k(\mathcal{Q}) \in \ell^1(\mathbb{Z}_+^*)$*, then* $\mathcal{Q}$ *is nuclear, and, if* $a_k(\mathcal{Q}) \in \ell^q(\mathbb{Z}_+^*)$ *for some* $q \in (0, 1]$*, then* $\mathcal{Q}$ *is* q*-nuclear.*

Using Pietsch's result, show that if $a_k(\mathcal{Q}) \in \ell^p(\mathbb{Z}_+^*)$ *for* $0 < p < \infty$ *then for any* $q \in (0, 1]$ *there exists* $\mathcal{N} = \mathcal{N}(p, q) < \infty$ *so that* $\mathcal{Q}^\mathcal{N}$ *is* q*-nuclear.*

Exercise 4. *Consider the matrix* $F = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ *acting on the two-torus.*

(*a*) *Prove that* F *is an area-preserving Anosov diffeomorphism.*

(*b*) *Give a complete proof of the bound on the essential spectral radius of* $\mathcal{L}\varphi(x) = (\varphi/|\det DF|) \circ F^{-1}(x)$ *acting on the Banach spaces* $\mathcal{B}^{t,s}$ *defined in the course for real numbers* $s < 0 < t$ *(or in Chapter 5.1 of the book).*

(*c*) *Compute the dynamical determinant*

$$d_{F^{-1},g}(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x:F^m(x)=x} \frac{\prod_{k=0}^{-(m-1)} g(F^k(x))}{|\det(\mathbb{1} - DF^{-m}(x))|}$$

and the dynamical zeta function

$$\zeta_{F^{-1},g}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{F^m(x)=x} \prod_{k=0}^{-(m-1)} g(F^k(x))$$

of F^{-1} *for the weight* $g = 1/|\det DF| \circ F^{-1}$*. Deduce from this and the results of the course that*

$$\int \varphi \circ F^n \psi dx - \int \varphi dx \int \psi dx$$

decays exponentially for C^α *functions* φ *and* ψ*, with* $\alpha > 0$*. Give an upper bound on the rate of mixing.*

2 Inverse problems

Exercise 5 (Computing symbols I)**.** *Consider the case of* $\mathbb{R}^2$*, recall*

$$I_0 f(\theta, s) = \int_{x\cdot\theta=s} f.$$

We take the adjoint with respect to the usual measure $d\theta ds$ *on* $\mathbb{S}^1 \times \mathbb{R}$*. Prove*

$$\frac{1}{2}\Delta^{1/2} I_0^* I_0 = \mathbb{1}.$$

(think of Fourier transforms).

For the rest of the exercises, we take some Riemannian surface with boundary (M, g).

Exercise 6. *Verify that strict infinitesimal convexity implies the convexity of the interior of* M*. In other words, assuming that the boundary is strictly convex —* $\mathrm{II} > 0$ *— show that there are no geodesics of* M *tangent to its boundary, and not reduced to a point.*

Exercise 7 (Integrating the flow). *Recall* $\Pi = I^*I$, *and*

$$R_\pm f = \pm \int_0^{\pm\infty} f \circ \varphi_t dt.$$

Assume that $f \in C^\infty(SM)$.

(a) *Check that*

$$\Pi f = (R_+ - R_-)f.$$

(b) *Deduce that* $f \in \ker I$ *if and only if there exists* $u \in C^\infty(SM \setminus \partial_0 SM) \cap C^0(SM)$, *vanishing at the boundary with* $-Xu = f$.

Exercise 8 (Computing some wavefront sets). *Recall Jared's talk and*

(a) *Prove that if* $u \in \mathcal{D}'(X \times Y)$ *and* $\pi : X \times Y \to Y$ *is the right projection,*

$$WF(\pi_* u) = d\pi(WF(u) \cap N^*X).$$

(b) *If* ψ_t *is a smooth flow, compute its wavefront set.*

(c) *Deduce the wavefront set of* Π.

Exercise 9 (Computing symbols II). *If* $\pi_0 : TM \to M$ *is the projection,* $I_0 = I\pi_0^*$ *and* $\Pi_0 = I_0^* I_0$.

(a) *Find the wavefront set of* Π_0.

(b) *When there are no conjugate points, check that* Π_0 *is a pseudor, and compute the principal symbol.(cut the operator into a smoothing part and another supported close to the diagonal).*

(c) *More generally, if* χ *is a cutoff around a transient trajectory, show* $I_0^*\chi^2 I_0$ *is a pseudor.*

Exercise 10 (Applications in the non-trapping case). *When there are neither conjugate points, nor trapped geodesics,*

(a) *Show that functions in* $\ker I_0 \cap L^1(M)$ *are smooth up to the boundary. (hint: imagine that you know that they have to vanish to all order at the boundary)*

(b) *Assuming that* I_0 *is injective, show that* I_0^* *is surjective from* $H^{s-1/2}(\partial SM)$ *to* $H^s(M)$ *(provided* $s > 1/2$*).*

Exercise 11. *Without assumption on conjugate points, show that*

$$I_0^* I_0 : H^{-1/2}_{comp}(M) \to H^{1/2}_{loc}(M).$$