BREVIARY FOR THE WEYL QUANTIZATION ON COMPACT MANIFOLDS

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Abstract. In this note, we review some properties of the Weyl quantization on compact manifolds. The purpose is to explain exactly which properties valid on $\mathbb{R}^n$ survive by the use of charts.

The Weyl quantization has particularly nice properties, which justify its use. Originally defined on $\mathbb{R}^n$ in a non-ambiguous way, one can use charts to obtain a corresponding quantization on a compact manifold. This process is arbitrary, because it depends on the choice of charts. However, several properties remain true independently of this choice.

We will review these properties, and give proofs or sketches of proofs. These are basic facts, and already well established, and available in several textbooks. However, here the point is present them in a condensed form. We hope that we can convince other members of the community that they are more simple than might be imagined.

Theorem 1. *Let M be a compact manifold without boundary of dimension n. Let $d\mu$ be a smooth volume form on M. We can build a semi-classical Weyl quantization* Op*, which takes functions $a \in C_c^\infty(T^*M)$ and gives smoothing operators $\mathrm{Op}(a)$ depending on a small parameter $h > 0$. We have the following properties:*

1. *For any other quantization Op' obtained in the same fashion, $\mathrm{Op}(a) = \mathrm{Op}'(a) + \mathcal{O}(h^2)$.*
2. *If a is real valued, $\mathrm{Op}(a)$ is symmetric.*
3. *If $b \in C_c^\infty(T^*M)$, there is a third function $c \in C_c^\infty(T^*M)$ such that*
$$\mathrm{Op}(c) = \mathrm{Op}(a)\,\mathrm{Op}(b) + \mathcal{O}(h^\infty),$$
where the remainder is smoothing, and
$$c = ab + \frac{h}{2i}\{a, b\} + \mathcal{O}(h^2).$$
4. *We also obtain*
$$[\mathrm{Op}(a), \mathrm{Op}(b)] = \frac{h}{i}\,\mathrm{Op}(\{a, b\}) + \mathcal{O}(h^3).$$

(5) *We have the bound*

$$\|\operatorname{Op}(a)\|_{L^2\to L^2} = \|a\|_{L^\infty} + \mathcal{O}(h).$$

(6) *Assume that a is real valued, and $a \geq 0$. Then as a quadratic form on L^2,*

$$\operatorname{Op}(a) \geq -Ch.$$

This is the sharp Gårding inequality*.*

(7) *Assume that a is real valued, and $a \geq 0$. Then as a quadratic form on L^2,*

$$\operatorname{Op}(a) \geq -Ch^2.$$

The constant C can be estimated using only the fourth derivative of a. This is the Fefferman-Phong inequality*.*

A few remarks are in order:

- It is crucial here that we have fixed a volume form. If one changes the volume form, property (1) is not true anymore. This is why the Weyl quantization is sometimes presented using the half-densities. However, in practical problems, there often is a reference volume form.
- When considering pseudo-differential operators acting on vector bundles, one has to choose local bases for the vector bundle. When changing bases, one changes the quantization by a $\mathcal{O}(h)$ instead of $\mathcal{O}(h^2)$, so there is no reason that one should get a lower bound of size $\mathcal{O}(h^2)$ from a purely symbolic condition. As a consequence, unless there is a special geometric structure on the bundle, only the Sharp Gårding inequality is available for bundles.
- Even for functions, the sharp Gårding inequality is often used instead of the Fefferman-Phong inequality. There is nothing to be gained by doing so.
- As we will see, the proof of the Fefferman-Phong inequality is almost entirely based on analysis on the principal symbol. One does not need to use the anti-Wick quantization, or fumble with heavy quantization formulae.

We also discuss usual symbol classes.

1. Formulae for the Weyl quantization in the euclidean space

Let us recall the formula for the Weyl quantization on $\mathbb{R}^n$. Given $a \in C_c^\infty(\mathbb{R}^{2n})$, $\mathrm{Op}_{\mathbb{R}^n}(a)$ is the operator whose kernel is

$$K(x, x') := \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-x', \xi\rangle} a\left(\frac{x+x'}{2}, \xi\right) d\xi.$$

From the formula, we directly get that when a is real valued, $K(x', x) = \overline{K(x, x')}$, so that $\mathrm{Op}_{\mathbb{R}^n}(a)$ is symmetric. More generally, $\mathrm{Op}_{\mathbb{R}^n}(\overline{a})$ is formally the adjoint of $\mathrm{Op}_{\mathbb{R}^n}(a)$.

The crucial formula is the formula for the product. Given a, b two smooth functions, formally, $\mathrm{Op}_{\mathbb{R}^n}(a)\,\mathrm{Op}_{\mathbb{R}^n}(b) = \mathrm{Op}_{\mathbb{R}^n}(c)$ with

$$c(x, \xi) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{4n}} e^{2\frac{i}{h}(\langle u, v\rangle - \langle u', v'\rangle)} a(x+u, \xi+v') b(x+u', \xi+v).$$

We can apply stationary phase to get that

$$c(x, \xi) = ab(x, \xi) + \frac{h}{2i}\{a, b\} + \mathcal{O}(h^2),$$

(provided a and b have the required regularity). We also find that outside of Ω the support of ab, c is $\mathcal{O}(h^\infty)$, decaying as $d((x, \xi), \Omega)^{-\infty}$ (for the usual metric on $\mathbb{R}^{2n}$).

2. Building the quantization: change of coordinates

We cover M by a finite set of U_i's, with coordinate charts $x_i : U_i \to \mathbb{R}^n$ such that $(x_i)_* d\mu = dx$ is the standard volume form in $\mathbb{R}^n$. These are called *isochore coordinates*. We choose χ_i's and η_i's such that

$$\sum \chi_i^2 = 1, \;\; \chi_i \eta_i = \chi_i, \;\; \sum \eta_i^2 \leq 2,$$

and $\chi_i, \eta_i \in C_c^\infty(U_i)$. We write

$$\chi^i(x) = \chi_i(x_i^{-1}(x)), \quad \text{and} \quad \eta^i(x) = \eta(x_i^{-1}(x)).$$

We can define pushforwards in the following fashion: for $a \in C_c^\infty(T^*M)$,

$$a_i(x, \xi) := ((x_i)_* a)(x, \xi) = a(x_i^{-1}(x), d_x(x_i^{-1})^T \xi).$$

Then a_i is a smooth function in $T^*(x_i(U_i)) \subset T^*\mathbb{R}^n = \mathbb{R}^{2n}$, and $\eta^i a_i \in C_c^\infty(\mathbb{R}^{2n})$. Given $f \in C^\infty(M)$, we let

$$\mathrm{Op}(a) f(x) := \sum_i (x_i)^* \left[\chi^i \,\mathrm{Op}_{\mathbb{R}^n}((\eta^i)^2 a_i)[(x_i)_* \chi_i f]\right](x),$$

We will seek to prove that this is a quantization later on. For now, we will determine how much the choices we have made influence the result.

First off, if we change the η_i keeping the properties above, then, we change the operator by a smoothing $\mathcal{O}(h^\infty)$ operator.

Next, consider $\tau : \mathbb{R}^n \to \mathbb{R}^n$ a *conservative* diffeomorphism, which is the identity outside of a compact set, and $a \in C_c^\infty(T^*\mathbb{R}^n)$. What is $\tau^* \operatorname{Op}_{\mathbb{R}^n}(a)$? The kernel of that operator is

$$K(x,x') = K_{\operatorname{Op}_{\mathbb{R}^n}(a)}(\tau(x),\tau(x'))$$

This is

$$\frac{1}{(2\pi h)^n}\int e^{i\langle \tau(x)-\tau(x'),\xi\rangle/h} a((\tau(x)+\tau(x'))/2,\xi)d\xi.$$

We would like to write this as $\operatorname{Op}_{\mathbb{R}^n}(b)$. We have an explicit formula for b:

$$b(x,\xi) = \int_{\mathbb{R}^n} e^{-i\langle u,\xi\rangle/h} K(x+u/2, x-u/2)du.$$

This can be rewritten in the form

$$\frac{1}{(2\pi h)^n}\int e^{i\Phi(x,\xi;u,\xi')/h} c(x,\xi;u,\xi')dud\xi.$$

The phase is

$$\Phi = -\langle u,\xi\rangle + \langle \tau\Big(x+\frac{u}{2}\Big) - \tau\Big(x-\frac{u}{2}\Big),\xi'\rangle = \langle u, d_x\tau^T\xi' - \xi\rangle + O(u^3\xi').$$

and the symbol

$$c = a\left(\frac{\tau(x+u/2)+\tau(x-u/2)}{2},\xi'\right).$$

In particular, there is a unique stationary point at $\xi' = (d_x\tau^T)^{-1}\xi$, $u=0$. At that point, the mixed determinant $\det(\partial_u\partial_{\xi'}\Phi)$ equals 1 because τ is conservative. We will give a few words on stationary phase in section 3. For now, since every function is compactly supported, there is no problem of convergence, and we use usual results. We can thus rewrite the integral in the following way: $(v = d_x\tau^T\xi' - \xi)$

$$\frac{1}{(2\pi h)^n}\int e^{i\langle u,v\rangle/h} f(u,v)dudv \sim \sum_{k\geq 0}(ih)^k(\partial_u\partial_v)^k f(0,0).$$

where

$$f = e^{\frac{i}{h}u^3\rho(u)(d_x\tau^{T-1}(\xi+v))} a(\tau(x) + \frac{d^2\tau(u,u)}{4} + O(u^4), d_x\tau^{T-1}(\xi+v)).$$

We conclude that

$$b(x,\xi) = a(\tau(x), d_x(\tau^{-1})^T\xi) + O(h^2).$$

Now, introduce other charts sets V_j, $\tilde{x}_j$, and corresponding $\tilde{\chi}_j$, $\tilde{\eta}_j$, $\tilde{\mathrm{Op}}$:

$$\tilde{\mathrm{Op}}(a) = \sum \tilde{\chi}_j \tilde{\mathrm{Op}}_j(\tilde{\eta}_j^2 a)\tilde{\chi}_j$$

(here $\tilde{\mathrm{Op}}_j = \tilde{x}_j^* \mathrm{Op}_{\mathbb{R}^n}$). We can write this as

$$\tilde{\mathrm{Op}}(a) = \sum_{i,j} \chi_i \tilde{\chi}_j \tilde{\mathrm{Op}}_j(\tilde{\eta}_j^2 a)\tilde{\chi}_j \chi_i$$

Up to smoothing $O(h^\infty)$ remainders, this is

$$\sum_{i,j} \chi_i \tilde{\chi}_j \tilde{\mathrm{Op}}_j(\eta_i^2 \tilde{\eta}_j^2 a)\tilde{\chi}_j \chi_i$$

Now, each $\eta_i^2 \eta_j^2 a$ is supported in the intersection of V_j with U_i, so we can change $\tilde{\mathrm{Op}}_j$ for Op_i, and by the computations above,

$$\sum_{i,j} \chi_i \tilde{\chi}_j \mathrm{Op}_i(\eta_i^2 a)\tilde{\chi}_j \chi_i + O(h^2).$$

Now recall that

$$\begin{aligned}
\chi \mathrm{Op}_{\mathbb{R}^n}(a)\chi &= \mathrm{Op}_{\mathbb{R}^n}(a\chi^2 + \frac{h}{2i}\{\chi, a\}\chi + \frac{h}{2i}\{a\chi, \chi\} + \mathcal{O}(h^2)) \\
&= \mathrm{Op}_{\mathbb{R}^n}(a\chi^2) + \mathcal{O}(h^2).
\end{aligned}$$

and we get that the sum before is equal to

$$\sum_i \chi_i \mathrm{Op}_i(\sum_j \eta_i^2 a \tilde{\chi}_j^2)\chi_i + O(h^2) = \mathrm{Op}(a) + O(h^2).$$

Conclusion : the Weyl quantization (with respect to a fixed volume form) is defined up to a $\mathcal{O}(h^2)$ remainder.

3. Symbol classes: exotic calculus without second microlocalization

Obviously, one wants to quantize symbols that are not only compactly supported. To this end, one needs to define *symbol classes*, which are classes of functions behaving in some particular way as $\xi \to \infty$ in the fibers. On manifolds, the simplest class is the *Kohn Nirenberg class.* We write $a \in S^m(T^*M)$ (for $m \in \mathbb{R}$) if in local coordinates

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}. \tag{1}$$

This does not depend on the choice of a metric on M as we will see in a few paragraphs. The type of integrals that we are led to compute at every turn take the form

$$\int e^{i\langle x,\xi\rangle/h} a(x,\xi) dx d\xi.$$

To estimate the contribution to the integral of the part far away from the stationary point, one introduces a cutoff χ supported away from 0, and tries to estimate the integral

$$I = \int e^{i\langle x,\xi\rangle/h} \chi(x,\xi) a(x,\xi) dx d\xi.$$

Then, if X is a vector field on $\mathbb{R}^{2n}$ such that $X\langle x,\xi\rangle \neq 0$ for $(x,\xi) \neq 0$, one does a series of integration by parts in the following fashion

$$I = \left(\frac{h}{i}\right)^k \int e^{i\langle x,\xi\rangle/h} L_X^k a,$$

with

$$L_X a = \frac{Xa}{X\langle x,\xi\rangle} - a\frac{X(X\langle x,\xi\rangle)}{(X\langle x,\xi\rangle)^2}.$$

This process will eventually conclude if after a finite number of integration by parts, the integrand becomes integrable. Obviously, one does not need to take only one vector field. In the particular case of the Kohn-Nirenberg symbols, it suffices to take

$$X = \xi\partial_x + x\partial_\xi.$$

One can imagine that other conditions than (1) can be imposed, and that one can then choose some vector fields adapted to the problem. This is the issue of the Weyl-Hörmander calculus, exposed for example in [Ler10]. The class of symbols $S^0(T^*M)$ can be seen as the set of functions on T^*M that are C^∞ bounded with respect to the metric on T^*M

$$g_{KN} := dx^2 + \frac{d\xi^2}{1+\xi^2}.$$

(To define this properly, one actually has to pick a metric on M, and then use the corresponding vertical/horizontal bundle given by the Levi-Civita connection.) Then, the idea is that one can replace this metric by any metric satisfying a particular set of assumptions.

Coming back to the Kohn-Nirenberg class, when one does a change of coordinates, it is necessary that the coordinate change interact in a "nice way" with the symbolic estimate. In practice, this means that the classes $S^m(T^*M)$ do not depend on the choice of coordinates. This can be seen using the formula:

$$\begin{aligned}\partial_x \tau^* a &= \partial_x \left[a(\tau(x), d_x(\tau^{-1})^T \xi)\right] \\ &= \partial_x a \cdot d_x \tau + \partial_\xi a \cdot \partial_x((\partial_x \tau)^{-1})^T \xi.\end{aligned}$$

From this, we observe that a more general class of symbols is as easily defined. We write $a \in S^m_\epsilon(T^*M)$ iff

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}(\epsilon/h)^{(|\alpha|+|\beta|)/2}\langle\xi\rangle^{m-|\beta|}.$$

or also $a \in S^m_{\epsilon,0}(T^*M)$ iff

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha,\beta}(\epsilon/h)^{(|\alpha|+|\beta|)/2}\langle\xi\rangle^{m+|\alpha|/2-|\beta|/2}.$$

Again, this is invariant under change of coordinates, and such symbols are called “exotic”. Then, if $a, b \in S^0_\epsilon(T^*\mathbb{R}^n)$ are compactly supported in x, we get that

$$\mathrm{Op}_{\mathbb{R}^n}(a)\,\mathrm{Op}_{\mathbb{R}^n}(b) = \mathrm{Op}_{\mathbb{R}^n}(c) + O(\epsilon^\infty \Psi_\epsilon^{-\infty})$$

with

$$c = ab + \frac{h}{2i}\{a, b\} + \cdots + \mathcal{O}(\epsilon^n \langle\xi\rangle^{-n}).$$

If a, b were in $S^0_{\epsilon,0}(T^*\mathbb{R}^n)$, then the remainder would be $\mathcal{O}(\epsilon^n)$, with no decay in ξ.

Another useful remark is that since $\langle\xi\rangle$ is *slowly varying* with respect to g_{KN} — $|\nabla\langle\xi\rangle| \leq C\langle\xi\rangle$ — one can use $\langle\xi\rangle$ as an ϵ parameter. This means that statements about S^m_ϵ, as a rule of thumb, also apply to $S^m_{\epsilon,0}$, which we can see as “$S^m_{\epsilon\langle\xi\rangle}$”.

The process of second microlocalization corresponds to wanting to quantize symbols that satisfy a condition of the type

$$|\partial_x^\alpha \partial_\xi^\beta a| \leq C(h/\epsilon)^{-|\alpha|},$$

with a loss of powers of h only differentiating in one direction. To be able to do this, one needs to go into much more technical details — which in some cases can be seen to be equivalent to taking a metric not equivalent to g_{KN} to estimate symbol norms.

4. Symbolic calculus on manifolds

We consider now $a, b \in C^\infty_c(T^*M)$ two symbols, and the product

$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \sum_{i,j} \chi_i\, \mathrm{Op}_i(a\eta_i^2)\chi_i\chi_j\, \mathrm{Op}_j(b\eta_j^2)\chi_j.$$

Since pseudo-differential operators are pseudo-local, we can choose ψ_j's such that $\chi_j\psi_j = \chi_j$ and $\psi_j\eta_j = \psi_j$. Then

$$\operatorname{Op}(a)\operatorname{Op}(b) = \sum_{i,j} \psi_i\chi_i \operatorname{Op}_i(a\eta_i^2)\chi_i\chi_j \operatorname{Op}_j(b\eta_j^2)\chi_j\psi_i + \mathcal{O}(h^\infty),$$

with a smoothing remainder. By the computations above,

$$\chi_i\chi_j \operatorname{Op}_j(b\eta_j^2)\chi_j\psi_i = \chi_i \operatorname{Op}_i(b\chi_j^2\eta_i^2 + \mathcal{O}(h^2))\psi_i + \mathcal{O}(h^\infty).$$

We deduce that

$$\begin{aligned}\operatorname{Op}(a)\operatorname{Op}(b) &= \sum_{i,j} \psi_i \operatorname{Op}_i(ab\chi_i^2\chi_j^2 + \frac{h}{2i}\{a\chi_i^2, b\chi_j^2\} + \mathcal{O}(h^2))\psi_i + \mathcal{O}(h^\infty),\\ &= \sum_i \psi_i \operatorname{Op}_i(ab\chi_i^2 + \frac{h}{2i}\{a\chi_i^2, b\} + \mathcal{O}(h^2))\psi_i + \mathcal{O}(h^\infty)\end{aligned}$$

Then, one can insert $1 = \sum \chi_j^2$ on both sides on the sum, to obtain points (3) and (4) of the theorem.

5. Boundedness result and Sharp Gårding

Having a nice algebraic structure for our operators would be fruitless if we could not make them act on functional spaces. Fortunately, they act as bounded operators on Sobolev spaces. The first observation is the following *Schur criterion*. If A is an operator mapping $C^\infty(M)$ to distributions on M, with kernel $K(x, x')$, if

$$\|K\|^2 := \sup_x \|K\|_{L^1(dx')} \sup_{x'} \|K\|_{L^1(dx)} < +\infty,$$

then A is bounded on $L^2(M)$, with norm $\|A\| \leq \|K\|$. Then, we obtain that if $a \in S_\epsilon^{-N}(T^*M)$ for N large enough, the Schur criterion is satisfied. The next step is that for $a \in S_{\epsilon,0}^{-N/2}(T^*M)$

$$\|\operatorname{Op}(a)u\|^2 = \langle \operatorname{Op}(a)^* \operatorname{Op}(a)u, u\rangle = \langle \operatorname{Op}(|a|^2 + \mathcal{O}(hS^{-N}))u, u\rangle.$$

By induction, we obtain the boundedness result for all symbols in $S_{\epsilon,0}^{-\delta}(T^*M)$, with $\delta > 0$.

To obtain the boundedness for operators in $\Psi_{\epsilon,0}^0$, the same argument does not apply because we do not gain powers of ξ in asymptotic expansions. However, we can use the Cotlar-Stein argument, which is just another avatar of the Schur Criterion. Assume that $A = \int A(z)dz$. Then

$$\|A\|^2 \leq \sup_z \int \|A(z)A(z')^*\|^{1/2}dz' \times \sup_z \int \|A(z)^*A(z')\|^{1/2}dz'.$$

Using non-stationary phase, one finds that if a is supported at a distance $\mathcal{O}(h/\epsilon\langle\xi\rangle)^{1/2}$ of $z \in T^*M$ and b is supported at the same distance of $z' \in T^*M$, then $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathcal{O}(h^\infty d_{KN}(z,z')^{-\infty})$ is smoothing.

This suggest to pick a partition of unity $1 = \int \chi_z dz$, where each. χ_z is supported at $\mathcal{O}(h/\epsilon\langle\xi\rangle)^{1/2}$ of $z \in T^*M$. We can do this in the form

$$\chi_z(\exp_z(u)) = F(\exp_x u)e^{-\epsilon|u|^2\langle\xi\rangle/h}\psi(|u|)$$

with $\psi \in C_c^\infty(\mathbb{R})$ equal to 1 around 0. This gives

$$\int \chi_z(z')dz \simeq F(z')\left(\frac{h}{\epsilon\langle\xi\rangle}\right)^n (1 + \mathcal{O}(\frac{h}{\epsilon\langle\xi\rangle})).$$

With that choice, we have $\int \chi_z(z')dz' \simeq 1$. Then, we are left to estimate the norm of $\mathrm{Op}(a\chi_z)$ indepently of z. By the Schur lemma, we find that

$$\|\mathrm{Op}(a\chi_z)\| \leq C\int |a(z')|\chi_z(z')dz' \leq C\|a\|_{L^\infty}.$$

(of course the full proof has more details).

Lemma 1. *Assume that $a \in S^0(T^*M)$ is real valued and positive. Then $b := \sqrt{a + h/\epsilon} \in S^0_\epsilon(T^*M)$. Additionally, we observe that derivatives of $\sqrt{a+h/\epsilon}$ behave as if it was in $\sqrt{h/\epsilon}S^0_\epsilon + S^0$.*

Proof. To prove this, one starts by observing that with respect to the metric g_{KN}, for $z \in T^*M$, $|u| \leq 1$,

$$0 \leq a(z') \leq a(z) + \langle\nabla a(z), u\rangle + C\|\nabla^2 a\|_{L^\infty(d(z,w)\leq 1)}|u|^2.$$

In particular,

$$|\nabla a(z)||u| \leq a(z) + C|u|^2.$$

So that optimizing $|u|$, either $a(z) \geq C'C$ for some constant $C' > 0$ depending only on g_{KN}, and C estimated using symbol norms for a, or

$$|\nabla a| \leq C\sqrt{a(z)}.$$

In the first case, $\sqrt{a+h/\epsilon} \in S^0$ locally uniformly. In the second case, one can use the Faà di Bruno formula for the higher order derivatives of $\sqrt{a+h/\epsilon}$. Absorbing all the first order derivatives of a with our bound, one gets the result:

$$(\sqrt{a+h/\epsilon})^{(k)} = \sum_{\ell=1}^{k}(a+h/\epsilon)^{1/2-\ell}\sum_{\substack{i_1+\cdots+i_\ell=k\\ i_j>0}} C_i\prod_{j=1}^{\ell} a^{(i_j)}(x).$$

So

$$|(\sqrt{a+h/\epsilon})^{(k)}| \leq C\sum_i (a+h/\epsilon)^{1/2-\ell_0/2-\ell+\ell_0}.$$

where $\ell_0 = \#\{i_j = 1\}$. Since $i_1 + \cdots + i_\ell = k$, $k \geq \ell_0 + 2(\ell - \ell_0)$, we deduce $(1 + \ell_0 - 2\ell)/2 \geq (1-k)/2$ and for $k \geq 1$,

$$|(\sqrt{a + h/\epsilon})^{(k)}| \leq C(\epsilon/h)^{(1-k)/2}.$$

□

In passing, we observe that if $a \in S^0_\epsilon(T^*M)$, and $\epsilon' > \epsilon$, then $\sqrt{a + \epsilon/\epsilon'} \in S^0_{\epsilon'}(T^*M)$.

Next, if $a \in S^1(T^*M)$ is positive, $a = \langle\xi\rangle a'$ with $a' \in S^0$. In particular, $\sqrt{a' + h/(\epsilon\langle\xi\rangle)} \in S^0_{\epsilon,0}(T^*M) + S^0(T^*M)$, and its derivatives behave as if it was in $(h/(\epsilon\langle\xi\rangle))^{1/2} S^0_{\epsilon,0}(T^*M) + S^0(T^*M)$. Then, we can write

$$\mathrm{Op}(a + h/\epsilon) = \mathrm{Op}(\sqrt{a + h/\epsilon})^2 + \mathcal{O}\left(\epsilon h \Psi^0_{\epsilon,0}\right).$$

Since a square is positive, we deduce the Sharp Gårding inequality directly. We also get the L^2 bound for $a \in S^0$ by considering the symbol $b = (\|a\|_{L^\infty} - a + h/\epsilon)^{1/2}$:

$$\mathrm{Op}(b)^2 = \|a\|^2_{L^\infty} - \mathrm{Op}(a) + h/\epsilon + \mathcal{O}(\epsilon h \Psi^{-1}_\epsilon) \geq 0.$$

6. The Fefferman-Phong inequality

We follow loosely the presentation of the proof given in [LM07]. To obtain the Fefferman-Phong result, one needs to use the fact that a smooth positive function can always be written as a sum of squares of smooth functions:

Lemma 2. *Let $f \in C^\infty_c(B(0,1))$ in $\mathbb{R}^{2n}$ be a real non-negative function with $\|d^4 f\|_{L^\infty(B(0,1))} \leq 1$. Let $\rho(x) = (f(x) + |d^2_x f|^2)^{1/4}$. Then one can find $f_1, \dots, f_\ell$, with ℓ depending only on the dimension, such that $f = f_1^2 + \cdots + f_\ell^2$, and each $f_i \in C^\infty_c(B(0,1))$, with $\|f_i\|_{C^k} \leq C_k \rho^{2-k}$. The constants $C_{0,1,2,3,4}$ only depend on the dimension.*

Proof. Assume that f is not the 0 function. We start with two preliminaries. From the Taylor expansion, we find that for some universal constants and $\ell = 0 \dots 4$,

$$|d^{(\ell)} f| \leq C\rho^{4-\ell}.$$

The next observation is that $|\nabla\rho| \leq C$, so that ρ has slow variation: $\rho(x + \lambda\rho(x)) \leq C_\lambda \rho(x)$. In particular, in a neighbourhood of size ρ, ρ does not vary too much.

First, assume that $f(x) > \epsilon\rho^4(x)$. In that case, $\|\sqrt{f}\|_{C^k} \lesssim C_\epsilon \rho^{2-k}$ for $k \geq 0$, and $|x' - x| < \rho(x)/C$.

On the other hand, if $f < \epsilon\rho^4$, then as ϵ becomes small enough,

$$|df| \leq C\epsilon^{1/2}\rho^3.$$

In that case, using

$$|df(x+u) - df(x) + d_x^2 fu| \leq C(\rho|u|^2 + |u|^3),$$

and $|d_x^2 f| > \sqrt{1-\epsilon}\rho^2$, taking $|u| = \rho\alpha$, with $< C\epsilon^{1/2} < \alpha < 1/C'$ with C' fixed large enough, we get that $d_x^2 fu$ can be larger (choosing the right vector direction) than $df(x)$ and $C(\rho|u|^2 + |u|^3)$. In particular, there is a point x' with $|x - x'| < \rho/C$ such that $df(x') = 0$. At that point, there is a direction u_0 such that $|d_{x'}^2 fu_0| > \rho^2/2$.

We deduce from the Local Inversion Theorem that we can find a locally smooth function X on $u_0^\perp$ such that

$$\langle \nabla f(x' + w + X(w)u_0), u_0 \rangle = 0,$$

for all $w \in u_0^\perp$ and $|w| < \rho/C$. Then

$$\begin{aligned} f(x' + w + (a + X(w))u_0) &= f(x' + w + X(w)u_0) \\ &\quad + a^2 \int_0^1 \langle d^2_{x'+w+(tX(w)+(1-t)a)u_0} fu_0, u_0 \rangle dt \end{aligned}$$

In this form, f is locally written as the sum of a square and a non-negative function depending on a strictly smaller number of variables:

$$f = \tilde{f} + f_1^2$$

We have that both $\|X\|_{C^k}$ and $\|(\int d^2 f)^{1/2}\|_{C^k}$ are of order ρ^{1-k} for $k \geq 1$. In particular, we have that $\|d^4 \tilde{f}\| \simeq 1$, and $\|f_1\|_{C^k} \lesssim \rho^{2-k}$ for $k \geq 1$ (again for $|x - x'| < \rho(x)/C$).

Proceeding by induction on the dimension, at each x for which $\rho \neq 0$, we can find $f_1, \ldots, f_{k(2n)}$ defined in a neighbourhood of size ρ, with $\|f_i\|_{C^k} \lesssim \rho^{2-k}$ for $k \geq 0$, and $f = f_1^2 + \cdots + f_{k(2n)}^2$.

Now, to finish the proof, we need to know that we can cover the space by balls of the type $B(x, \epsilon\rho(x))$ so that there

(1) There is a number $N(n)$ such that there never more than $N(n)$ balls with non empty intersection.
(2) There is a corresponding partition of unity $1 = \sum \chi_i$ such that $\sup \rho^k \|\chi_i^{(k)}\| < \infty$ for all $k \geq 0$.

This is contained in the Theorem 1.4.10 in [Hör03] □

Now, we can adapt this local result to the manifold:

Lemma 3. *Let $a \in S^2(T^*M)$ be real non-negative. Let*

$$\rho(z) = \langle \xi \rangle^{-1/2} (a(z) + \langle \xi \rangle^{-2} |\nabla_z^2 a|^2)^{1/4},$$

(with respect to g_{KN}) Then there are $b_1, \ldots, b_k$ smooth functions such that $a = b_1^2 + \cdots + b_k^2 + \mathcal{O}(h^2 S^0_{\epsilon,0})$. We have that

$$b_i = \langle \xi \rangle b_i',$$

and b_i' is supported for $\rho \geq (h/\epsilon\langle\xi\rangle)^{1/2}$, with

$$|\nabla^k b_i| \leq C\langle\xi\rangle(1+\rho^{2-k}).$$

for $k \geq 2$

Proof. We apply the lemma above to $a' = \langle\xi\rangle^{-2}a$, in exponential charts for the metric g_{KN}. The local ρ for a' and the global ρ coincide up to a multiplicative constant, so the estimate are conserved by this procedure. By this process, we obtain an infinite number of $\tilde{b}_i$, however, since the overlap of the balls can be chosen to be finite uniformly bounded, we can regroup those $\tilde{b}_i$'s in packets of disjoint $\tilde{b}_i$'s. In this way, we obtain $\langle\xi\rangle^{-2}a = \tilde{b}_1^2 + \cdots + \tilde{b}_k^2$. The next step is to introduce a cutoff

$$\eta := \chi(\rho(\epsilon\langle\xi\rangle/h)^{1/2}),$$

where $\chi \in C_c^\infty([0,1[)$, constant equal to 1 around 0, and let $b_i = \langle\xi\rangle\tilde{b}_i\eta$. We have for $k \geq 0$,

$$|\nabla^k\eta| \leq (\epsilon\langle\xi\rangle/h)^{k/2} \sim \rho^{-k},$$

so that $a(1-\eta) \in (h/\epsilon)^2 S^0_{\epsilon,0}$, and the announced estimate holds for b_i. ☐

Now, we can write down:

$$\mathrm{Op}(a) = \mathrm{Op}(b_1)^2 + \cdots + \mathrm{Op}(b_k)^2 + \mathcal{O}((h/\epsilon)^2\Psi^0_{\epsilon,0} + h^2\Psi^0_{\epsilon,0})$$

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