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Les résonances du laplacien sur les variétés à pointes

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❈

Résumé

Cette thèse a pour objet l'étude des résonances du laplacien sur les variétés à pointes. Ce sont des variétés dont les bouts sont des pointes hyperboliques réelles. Ces objets ont été introduits par Selberg pour les surfaces à pointes de courbure constante dans les années 50. Leur définition a ensuite été étendue en courbure variable par Lax et Phillips.

Les résonances sont les pôles d'une famille méromorphe de fonctions propres généralisées du laplacien. Elles sont associées au spectre continu du laplacien. Pour analyser ce spectre continu, plusieurs directions de recherche sont explorées ici.

D'une part, on obtient des résultats sur la localisation de ces résonances. En particulier, si la courbure est négative, on montre que pour un ensemble générique de métriques, les résonances se séparent en deux ensembles. Le premier est contenu dans une bande près du spectre continu. L'autre partie est composée de résonances qui s'éloignent du spectre. Ceci laisse une zone de taille log sans résonance.

D'autre part, on étudie les mesures microlocales associées à certaines suites de paramètres spectraux. En particulier, on montre que pour des suites de paramètres spectraux qui s'approchent du spectre, mais pas trop vite, la mesure microlocale associée est nécessairement la mesure de Liouville. Cette propriété est valable quand la courbure de la variété est négative.

mots clés Variétés à pointes. Analyse microlocale. Courbure négative. Mesures microlocales. Paramétrice semi-classique. Déterminant de scattering. Loi de Weyl.

Abstract

The resonances of the Laplace operator on cusp manifolds.

In this thesis, we study the resonances of the Laplace operator on cusp manifolds. They are manifolds whose ends are real hyperbolic cusps. The resonances were introduced by Selberg in the 50's for the constant curvature cusp surfaces. Their definition was later extended to the case of variable curvature by Lax and Phillips.

The resonances are the poles of a meromorphic family of generalized eigenfunctions of the Laplace operator. They are associated to the continuous spectrum of the Laplace operator. To analyze this continuous spectrum, different directions of research are investigated.

On the one hand, we obtain results on the localization of resonances. In particular, if the curvature is negative, for a generic set of metrics, they split into two sets. The first one is included in a band near the spectrum. The other is composed of resonances that are far from the spectrum. This leaves a log zone without resonances.

On the other hand, we study the microlocal measures associated to certain sequences of spectral parameters. In particular we show that for some sequences of parameters that converge to the spectrum, but not too fast, the associated microlocal measure has to be the Liouville measure. This property holds when the curvature is negative.

keywords Cusp manifolds. Microlocal analysis. Negative curvature. Microlocal measures. Semi-classical parametrix. Scattering determinant. Weyl law.

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Avant-propos

Laplacien, Mécanique Quantique, et Analyse microlocale

Étant donnée une variété riemannienne compacte (M, g) de dimension $d+1$, on peut former son opérateur de Laplace-Beltrami $-\Delta_g$. C'est un opérateur différentiel d'ordre 2, positif sur $L^2(M)$, qui généralise le Laplacien sur $\mathbb{R}^n$. Son spectre est discret et on ordonne ses valeurs propres $\lambda_0 = 0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \ldots$. Le problème de déterminer exactement les λ_i est extrêmement compliqué, et n'a de solution que dans des cas très particuliers. Par contre, si l'on renonce à obtenir des informations sur les valeurs propres individuelles, mais que l'on cherche des propriétés de moyenne, on arrive à de nombreux résultats, qui font intervenir des propriétés plus ou moins fines de la géométrie de (M, g). La plus élémentaire est la Loi de Weyl faible [Zwo12, Theorem 6.8]. Elle permet de compter le nombre de valeurs propres λ_i plus petites que T^2.

$$\#\left\{\lambda_i \leqslant T^2\right\} = \frac{vol(M)}{(4\pi)^{(d+1)/2}\Gamma((d+3)/2)}T^{d+1} + o(T^{d+1}). \qquad \text{(Loi de Weyl faible)}$$

En toute généralité, on peut améliorer ce résultat [Hör68, Sog93] pour obtenir

$$\#\left\{\lambda_i \leqslant T^2\right\} = \frac{vol(M)}{(4\pi)^{(d+1)/2}\Gamma((d+3)/2)}T^{d+1} + \mathcal{O}(T^{d}). \qquad \text{(Loi de Weyl forte)}$$

Cette nouvelle formule est optimale car il y a (au moins) un exemple de variété qui sature la borne sur le reste : la sphère $\mathbb{S}^{d+1}$. Dans ce cas, les valeurs propres ont des multiplicités qui peuvent être très élevées, et cela rend la fonction de comptage particulièrement irrégulière. Si on suppose que la courbure de la variété est strictement négative, on obtient [Bér77]

$$\#\left\{\lambda_i \leqslant T^2\right\} = \frac{vol(M)}{(4\pi)^{(d+1)/2}\Gamma((d+3)/2)}T^{d+1} + \mathcal{O}\left(\frac{T^d}{\log T}\right).$$

(Loi de Weyl avec reste de Bérard)

Il n'est pas clair que ce résultat soit optimal, et il existe à ce jour de nombreux travaux sur le sujet [JP07, JPT08]. Quand la courbure est négative, on peut aussi citer le théorème de Chazarain [Cha74] (précisé ensuite par Duistermaat-Guillemin [DG75]). Si $\hat{\varsigma} := \operatorname{Tr} e^{it\sqrt{\Delta}}$ (c'est une distribution en t),

$$\operatorname{singsupp} \hat{\varsigma} \subset \{0\} \cup \{ \text{ longueurs des géodésiques fermées sur } M \}. \qquad \text{(Ch)}$$

Pour prouver de tels résultats, il est nécessaire d'introduire un formalisme d'analyse microlocale. Faisons d'abord un détour par la physique.

Les postulats de la mécanique

Quand on veut décrire le mouvement d'un point matériel $x(t)$, de masse m, soumis à une force F, dans l'espace euclidien, on doit intégrer l'équation de Newton

$$m\ddot{x} = F. \qquad \text{(Loi de Newton)}$$

Si l'on considère maintenant la dynamique d'un objet qui n'évolue plus forcément dans l'espace euclidien, mais sous une dynamique conservative (i.e. sans phénomènes de frottements, par exemple), la Mécanique Classique postule que l'on peut en décrire le mouvement de la façon suivante. L'état de cet objet est représenté par un point ρ dans une variété symplectique (X, ω). Sur cette variété est définie une fonction H appelée *hamiltonien*, qui mesure l'énergie du point p, et le mouvement de p est donnée par le *flot hamiltonien* associé à H. Si f est une observable, autrement dit une fonction sur X, alors l'évolution de la valeur de f le long du mouvement est donnée par

$$\frac{Df}{Dt} = \{f, H\}. \qquad \text{(Loi de Hamilton)}$$

où $\{.,.\}$ est le crochet de Poisson sur X donné par la structure symplectique. La Mécanique Classique ne permet pas de rendre compte de certains phénomènes physiques (l'exemple le plus célèbre étant peut-être l'effet photoelectrique). La Mécanique Quantique a été développée pour combler ces failles. Ses postulats sont les suivants : un état physique peut être représenté par un point ψ dans la sphère unité d'un Hilbert complexe $\mathcal{H}$, sur lequel est défini un opérateur linéaire $\widehat{H}$ encore appelé *hamiltonien*. L'évolution de ce point est donnée par

$$i\hbar\frac{d}{dt}\psi = \hat{H}\psi. \qquad \text{(Équation de Schrödinger)}$$

La quantité $\hbar$ étant une constante de la physique, que l'on peut mesurer ($\hbar \simeq 1,05.10^{-34}m^2.kg.s^{-1}$). Les observables physiques sont désormais des éléments de $End(\mathcal{H})$, et si un tel endomorphisme A est donné, alors mesurer A sur ψ doit donner en moyenne

$$\langle A\rangle_\psi = \langle A\psi, \psi\rangle. \qquad \text{(Valeur moyenne d'une observable Quantique)}$$

On retrouve alors

$$i\hbar\frac{d}{dt}\langle A\rangle_\psi = \langle[A, \widehat{H}]\rangle_\psi. \qquad \text{(Loi d'Ehrenfest)}$$

Ce qui ressemble formellement à la Loi de Hamilton. Quand la particule quantique possède aussi une description classique, les fondateurs de la mécanique quantique on donné des *règles de quantification* qui permettent de faire le lien entre les deux formulations (par exemple, comment choisit-on le hilbert, et le hamiltonien ?). En toute généralité, les énoncer dépasse le cadre de cet opus ; c'est le point de départ de la théorie de la *quantification géométrique* (voir [BdMG81]).

Dans le cas où la particule est un point matériel dans une variété (M, g) (sans spin, par exemple), l'espace des phase X est le cotangent T^*M de M. Alors on peut prendre $\mathcal{H} = L^2(M)$. Si on veut quantifier une observable qui ne dépend que de la position, i.e $a \in C^\infty(M)$, le bon opérateur est $a(\hat{x})$, l'opérateur de multiplication par a sur $L^2(M)$. Pour quantifier l'impulsion, il faut prendre $\hat{p} = -i\hbar\nabla$. C'est un opérateur essentiellement auto-adjoint non-borné sur $\mathcal{H}$, et donc naturellement, pour quantifier les fonctions de l'impulsion, il faut prendre les fonctions de $-i\hbar\nabla$.

Dès que l'on commence à vouloir quantifier des observables qui dépendent simultanément de la position et de l'impulsion, on rencontre un problème. En effet, la famille des opérateurs $a(\hat{x})$ et $f(\hat{p})$ ne commutent pas. Il semble donc il y avoir un problème pour définir ce que serait $a(\hat{x}, \hat{p})$.

L'analyse microlocale

Dans le cas de $\mathbb{R}^n$, on sait que $-i\nabla$ correspond à la multiplication par ξ en Fourier. Autrement dit, les observables de positions s'écrivent facilement comme des opérateurs de multiplication, et les observables d'impulsion s'écrivent comme des opérateurs de multiplication en Fourier.

Naïvement, si $f \in L^2(M)$, et $a \in C^\infty(T^*M)$, on veut définir $a(x, -i\hbar\nabla)f$ comme

$$\mathscr{F}^{-1}\{a.\mathscr{F}f\} \tag{$\hat{\not{a}}$}$$

où $\mathscr{F}$ serait une « transformée de Fourier ». Cette expression ne semble pas avoir de sens. Pour pouvoir faire fonctionner un tel programme, on est donc obligé d'introduire des détails techniques. D'abord, comme M a de la géométrie, il n'y a pas de raison que M ait en général une transformée de Fourier globale (c'est seulement le cas quand M est localement symétrique, et que l'on dispose de la transformée de Fourier-Helgason). Voir [Hel70, Hel94]. Voir aussi [Zel86].

La première étape est de *découper* la variété. On se donne donc une fonction $\chi(x, x')$ sur $M \times M$, qui est supportée dans un petit voisinage de la diagonale $\{(x, x),\ x \in M\}$, et vaut 1 près de la diagonale. Autour de chaque point $x \in M$, on se donne une carte locale, ce qui permet de donner un sens à $\langle x - x', \xi\rangle$ pour (x, x') dans le support de χ. Alors on pose

$$\mathrm{Op}(a)f(x) := \int e^{i\langle x-x',\xi\rangle} a(x, \hbar\xi) f(x')\chi(x, x')dx'd\xi. \tag{$\hat{a}$}$$

C'est par cette formule que l'on donne un sens à l'équation ($\hat{\not{a}}$). En essayant de construire une théorie robuste pour de tels opérateurs sur $L^2(M)$, on se rend compte de plusieurs choses.

1. Tous les opérateurs différentiels sur M sont dans la classe des opérateurs que l'on vient de décrire. Si a est un polynôme en ξ sur T^*M d'ordre n, alors $\mathrm{Op}(a)$ est un opérateur différentiel d'ordre n sur M. De plus, si $\mathrm{Op}'(a)$ est une autre quantification de a, alors $\mathrm{Op}(a) - \mathrm{Op}'(a)$ est un opérateur différentiel d'ordre $n - 1$ sur M.

2. L'hypothèse $a \in C^\infty(T^*M)$ est trop faible pour avoir une théorie agréable. Il faut au moins que $a \in \mathcal{S}'(T^*M)$ soit tempérée. C'est le cas si, par exemple a admet un développement polyhomogène en ξ quand $|\xi| \to \infty$.

3. Pour que $\mathrm{Op}(a)$ agisse vraiment sur $L^2(M)$, ou au moins sur les espaces de Sobolev, et pas seulement sur $C^\infty(M)$, il faut imposer des conditions plus fortes sur a. Ces conditions sont rassemblées dans la familles des *estimées symboliques*. Il faut que a soit C^∞, et ait une croissance *raisonnable* en $|\xi| \to \infty$. Ces estimées sont toujours satisfaites par les fonctions à support compact, et aussi par les fonctions qui admettent un développement polyhomogène en ξ.

4. On a choisit une troncature χ puis des cartes. On pourrait penser a priori qu'il y a un choix plus malin, ou plus physique que les autres. En fait, si on prend un autre choix, on obtient une autre quantification Op', et si a est une observable classique, on peut trouver b une autre observable classique telle que $\mathrm{Op}(a) = \mathrm{Op}'(b)$, modulo un reste, qui est un opérateur négligeable (au sens de la théorie). De plus, on trouve (informellement) que $b = a + o(a)$. Ainsi, la classe des opérateurs ainsi obtenus ne dépend pas des choix que l'on a fait.

Les opérateurs obtenus de cette façon sont des *Opérateurs Pseudo-différentiels* (ou *pseudos*), et ils forment la classe des observables de position et d'impulsion que l'on manipule. C'est une algèbre d'opérateurs sur $L^2(M)$.

Si on se donne un pseudo $\mathrm{Op}(a)$, on veut construire la classe d'équivalence des symboles a' tels que $a'/a \to 1$ quand $\xi \to \infty$. Au prix de quelques détails, on peut le faire de façon raisonnable, et on obtient un objet qui ne dépend plus des choix menant à la construction de Op. C'est le *symbole principal*, et c'est un morphisme d'algèbre. Finalement, une quantification peut être vue comme une section de ce morphisme. Dans un problème pratique, il faut faire de la physique pour déterminer la «bonne quantification».

Ceci étant dit, l'utilité des pseudos dépasse largement le cadre de la mécanique quantique, et ils sont l'objet de la théorie de l'*analyse microlocale*. Pour une introduction, on peut se référer par exemple à [Sog93, Zwo12, GS94].

L'approximation semi-classique

Revenons à la dynamique d'une particule dans une variété riemannienne. Quand elle suit un mouvement *libre*, alors on peut choisir comme hamiltonien $\widehat{H} = -\hbar^2\Delta_g/2m$. L'étude spectrale du laplacien permet donc (en théorie) de connaître le mouvement d'une particule quantique libre dans une variété riemannienne M. Or selon les postulat de la Mécanique Classique, c'est le flot hamiltonien de $H_0(p) = |p|^2/2$, autrement dit, le flot géodésique φ_t de M, qui décrit l'évolution de l'état classique de la particule.

On peut faire le lien entre les deux descriptions. Une particule quantique possède une longueur d'onde, que l'on peut estimer comme $\lambda \sim \hbar/\langle|\hat{p}|\langle_\psi$. Quand on mesure une observable $\mathrm{Op}(a)$ où a satisfait des estimées symboliques, on est en train de mesurer des quantités qui varient à une échelle de longueur qui est commensurable à la taille L de la variété M. Si on prend une limite $\lambda/L \to 0$, on doit observer que la mécanique classique approxime bien la mécanique quantique.

Cela correspond en pratique à avoir $|\nabla\psi| \to \infty$. On peut reformuler cette idée de la façon suivante :

> *Le comportement des fonctions propres du laplacien doit pouvoir être relié à des propriétés du flot géodésique, dans la limite des grandes oscillations.*

Pour mettre ceci en pratique, on considère une suite de fonctions propres du laplacien $-h^2\Delta u_h = u_h$, où h est un petit paramètre. Heuristiquement, u_h doit osciller à une longueur d'onde $\lambda \sim h$. En particulier, la transformée de Fourier de sa restriction à un petit ouvert doit être supportée autour de $\xi \sim h^{-1}$. Pour observer un phénomène intéressant, on se donne $a \in C_c^\infty(T^*M)$ qui est supportée autour de $|\xi| = 1$, et on pose $a_h(x,\xi) = a(x, h\xi)$, puis $\mathrm{Op}_h(a) = \mathrm{Op}(a_h)$. On mesure alors

$$\langle \mathrm{Op}_h(a)u_h, u_h\rangle. \qquad \text{(Distributions de Wigner)}$$

On peut alors montrer que les valeurs d'adhérence de cette famille de distribution sur T^*M sont des mesures sur la cosphère unité S^*M, qui sont invariantes par le flot géodésique. Les distributions $a \mapsto \langle \mathrm{Op}_h(a)u_h, u_h\rangle$ sont appelées *distributions de Wigner* par analogie avec le cas de $\mathbb{R}^{d+1}$.

Autrement dit, en observant le comportement d'une particule de longueur d'onde $\sim h$, pour des observables supportées à des impulsions $\sim h^{-1}$, on obtient une limite non triviale, et cette limite correspond à des objets essentiels en mécanique classique. Il n'est pas inutile d'insister sur le fait que si on considère $\langle \mathrm{Op}_{h'}(a)u_h, u_h\rangle$ avec $h/h' \to 0$ ou $h/h' \to \infty$, la limite obtenue est toujours 0 (car a n'est pas supportée près de $\xi = 0$).

Par ailleurs, si a n'est pas supportée dans un voisinage de S^*M, on trouve

$$\mathrm{Op}_h(a)u_h = \mathcal{O}_{L^2}(h^\infty)\|u_h\|_{L^2}. \qquad \text{(Micro-Localisation des fonctions propres)}$$

Plus généralement, les procédés qui consistent à étudier les solutions d'une EDP qui oscillent à petite longueur d'onde h en les mesurant avec des opérateurs pseudo-différentiels dont le symbole est supporté à impulsion $\sim h^{-1}$ sont rassemblés dans l'analyse microlocale dite *semi-classique*. La famille $(\mathrm{Op}_h(a))_{h>0}$ est l'exemple type d'opérateur pseudo-différentiel semi-classique (ou *h-pseudo*).

On introduit aussi les *états lagrangiens*. Si S et χ sont des fonctions sur M, alors $u_h = \chi e^{iS/h}$ est une *famille semi-classique d'états lagrangiens* sur M. D'un point de vue physique, u_h représente une particule quantique dont la probabilité d'être au point x est $|\chi(x)|^2$, et dont la longueur d'onde est $\sim h$, et qui oscille dans la direction ∇S. Une propriété fondamentale des h-pseudos est qu'ils préservent les états lagrangiens au sens où

$$\mathrm{Op}_h(a)u_h(x) = a(x, d_x S)\chi(x)e^{iS(x)/h} + \mathcal{O}(h). \tag{1}$$

Dans ce contexte, on peut reformuler les résultats précédents (Loi de Weyl faible, Loi de Weyl forte, ...) comme des résultats qui relient le comportement du spectre de $\mathrm{Op}_h(a)$ autour de $z \in \mathbb{R}$ à la dynamique hamiltonienne de a dans les niveaux d'énergie $\{a(x,\xi) \simeq z\}$. Plus h est petit, plus les approximations que l'on obtient sont bonnes, ce qui revient à dire encore une fois que le lien entre spectre et géométrie est meilleur *à haute fréquence*.

Dans la suite de ce texte on n'utilisera que des h-pseudos ; pour alléger les notations on oubliera rapidement le «h» dans Op_h.

Chaos Quantique

Depuis au moins Poincaré, on sait que les systèmes dynamiques chaotiques présentent des comportements extrêmement variés et que cette richesse se reflète dans la structure de l'espace des mesures invariantes. Ce sont les travaux d'Anosov, Smale [Sma67] et de nombreux autres [PP90] qui ont permi de rassembler ces phénomènes dans la théorie des systèmes *uniformément hyperboliques*.

Les flots géodésiques de variétés compactes à courbure négatives en sont de très bons exemples : ils ont la propriété d'Anosov. Cela veut dire qu'il y a deux fibrés E^s et E^u dans TS^*M, et des constantes $C > 0$, $\lambda > 0$ telles que

$$TS^*M = \mathbb{R}\mathbf{X} \oplus E^s \oplus E^u \quad \text{et pour tout } t > 0, \quad \begin{cases} \|(d\varphi_t)_{|E^s}\| \leqslant Ce^{-\lambda t} \\ \|(d\varphi_{-t})_{|E^u}\| \leqslant Ce^{-\lambda t} \end{cases}$$

(Propriété d'Anosov)

Cela implique en particulier que le flot géodésique est exponentiellement mélangeant [Hop71, Rat87, Dol98, Liv04].

Étant donné l'importance des mesures invariantes pour l'étude dynamique du flot géodésique sur S^*M, la question qui vient naturellement est la suivante :

> *Quelle est la classe des mesures que l'on peut obtenir comme valeur d'adhérence de distributions de Wigner $\langle \mathrm{Op}_h(a)u_h, u_h\rangle$?* De telles mesures sont dites *semi-classiques.*

Dans le cas de la sphère $\mathbb{S}^{d+1}$ (qui est de courbure positive, et non chaotique), on peut trouver des suites de fonctions propres du laplacien qui se concentrent sur l'équateur. Autrement dit, l'ensemble des mesures semi-classique contient les mesures supportées sur une seule orbite périodique.

L'heuristique du *Chaos Quantique* est que ceci ne peut pas se produire quand le flot géodésique de la variété est chaotique, et que la seule mesure semi-classique est la mesure de Liouville. Autrement dit

$$\langle \mathrm{Op}_h(a)u_h, u_h\rangle \overset{?}{\underset{h\to 0}{\to}} \int_{S^*M} a d\mathscr{L} \qquad \text{(Équidistribution des fonctions propres)}$$

où $\mathscr{L}$ est la mesure de Liouville sur S^*M.

Le meilleur exemple d'application de ce principe est le suivant. Supposons que le flot géodésique de M soit ergodique. Alors effectivement, si on prend la suite complète des valeurs propres du laplacien (λ_n), avec $u_n := u_{h_n}$,

$$\frac{1}{n}\sum_{m=0}^{n}\left|\langle \mathrm{Op}_{h_m}(a)u_m, u_m\rangle - \int_{S^*M} a d\mathscr{L}\right|^2 \underset{n\to+\infty}{\to} 0 \qquad \text{(Ergodicité Quantique)}$$

Ce résultat est du à Schirel'man [Šni74], Y. Colin de Verdière [CdV85] et S. Zelditch [Zel87].

En conséquence de ceci, Z. Rudnick et P. Sarnak [RS94] ont formalisé la Conjecture d'Unique Ergodicité Quantique (QUE), qui consiste à dire que la propriété d'Équidistribution des fonctions propres est valable pour toute les suites de fonctions propres. C'est un problème qui a une trentaine d'année, apparaissant déjà dans [CdV85] ; pour le moment, les seules solutions ont été trouvées dans des cas très particuliers [Lin06, Sou10]. Le problème pour attaquer le cas général étant que l'on ne semble pas disposer des outils pour faire la différence entre le cas du Laplacien sur les variétés de courbure négative et des exemples pour lesquels la propriété de QUE n'est pas vérifiée [FNDB03]. Pour en savoir plus, on peut lire (par exemple) [Sar09].

Variétés non-compactes, résonances et états résonnants

On cherche à généraliser les propos qui précèdent à des variétés non compactes, ce qui peut se justifier autant d'un point de vue physique (toutes les particules ne vivent pas dans un compact), que d'un point de vue mathématique (il y a des variétés non compactes dont on voudrait pouvoir faire une théorie spectrale raffinée, voir plus loin). On est confronté à un certain nombre de difficultés. La première d'entre elles est de savoir dans quel type de variétés non-compactes on veut se placer.

Variétés Géométriquement finies

En toute généralité, les variétés non-compactes peuvent exhiber des pathologies très variées. Une approche est donc de déterminer les variétés pour lesquelles la théorie de la dynamique du flot géodésique est suffisemment riche. Un bon exemple est celui des *variétés géométriquement finies.*

Soit X une variété de *Hadamard pincée.* Autrement dit, X est simplement connexe, de courbure pincée entre $-k_{max}^2$ et $-k_{min}^2$, avec $0 \leqslant k_{min} \leqslant k_{max}$. On appelle *bord visuel* de X l'espace topologique $\partial_\infty X$ obtenu en quotientant T^*X par la relation

$$(x,\xi) \sim (x',\xi') \iff d(\varphi_t(x,\xi),\varphi_t(x',\xi')) \text{ est bornée pour } t>0. \qquad \text{(Bord visuel)}$$

On peut définir une topologie sur $\overline{X} = X \cup \partial_\infty X$ qui en fait un espace compact.

Soit Γ un groupe discret d'isométrie de X, agissant librement. Si $x \in X$, alors on appelle *l'ensemble limite* de Γ, $\Lambda_\Gamma = \overline{\Gamma.x} \cap \partial X$. On peut vérifier que ceci ne dépend pas de x. Le *cœur convexe* $C(\Gamma)$ de Γ est l'enveloppe convexe de Λ_Γ dans X. On dit que Γ, ou X/Γ, est *géométriquement fini* si $C(\Gamma)/\Gamma$ est de volume fini. Cette terminologie a été introduite par Bowditch [Bow95] (nous en reproduisons ici la condition (F5)).

Pour de telles variétés, on peut trouver une théorie robuste des systèmes dynamiques, particulièrement quand $k_{min} > 0$. On peut se référer au livre [PPS12], et consulter (par exemple) [Dal99, Dal00, DOP00].

Prolongement méromorphes de résolvantes

Si on se tourne maintenant vers le Laplacien, on constate que son spectre n'est plus uniquement constitué de spectre discret. Il y a alors une observation cruciale. Considérons le Laplacien sur $\mathbb{R}$. Son spectre est entièrement continu, constitué de $M^0 = \mathbb{R}^+$, de multiplicité 1. On peut en fait calculer explicitement le noyau de sa résolvante, $K(\lambda; x, y)$ grace à la transformée de Fourier :

$$(-\Delta-\lambda^2)^{-1}f(x) = \frac{1}{2\pi}\int_{T^*\mathbb{R}} \frac{1}{|\xi|^2-\lambda^2} u_\xi(x)\langle f, u_\xi\rangle d\xi \text{ pour } f \in C_0^\infty(\mathbb{R}),$$
(Résolvante de $-\Delta$ sur $\mathbb{R}$)

où les $u_\xi(x) = \exp\{ix\xi\}$ sont les *ondes planes*, et $\langle\ .\ ,\ .\ \rangle$ est le produit de dualité (hermitien, anti-linéaire à droite). On peut obtenir une telle formule sur n'importe quel espace qui supporte une transformée de Fourier-Helgason (voir [Hel70]). C'est bien sûr le cas des espaces euclidiens $\mathbb{R}^{d+1}$, mais plus généralement, des espaces symétriques, et en particulier l'espace hyperbolique réel.

Ce type de résultat a deux intérêts. D'abord, on retrouve une décomposition avec des fonctions propres *généralisées.* Ensuite, la formule ci-dessus ne fait a priori de sens que pour $\lambda \notin \mathbb{R}$, car la résolvante $(-\Delta-\mu)^{-1}$ n'est définie que pour $\mu \notin \mathbb{R}^+$. Par contre le membre de droite peut parfaitement être prolongé de $\{\Im\lambda > 0\}$ à $\mathbb{C}$ tout entier car on calcule

$$K(\lambda, x, y) = \frac{i}{2\lambda} e^{i|x-x'|\lambda}, \quad \Im\lambda > 0. \tag{2}$$

Autrement dit, $(-\Delta-\lambda^2)^{-1}$ peut être prolongée comme famille méromorphe d'opérateur de C_0^∞ dans C^∞ dépendant de $\lambda \in \mathbb{C}$. Encore une fois, ceci se produit pour une grand variété d'espaces symétriques [MV05, Str05].

On peut décrire les variétés géométriquement finies complètes comme des variétés compactes sur lesquelles on a recollé un certain nombre de *bouts.* À priori, ces bouts peuvent prendre des formes assez variées. Supposons que $M = M_0 \cup Z_1 \cup \cdots \cup Z_\kappa$, où M_0 est compacte à bord, et les bouts Z_i sont tous localement symétriques.

A partir de résultat très généraux sur les perturbations d'opérateurs auto-adjoints, de la théorie de la diffusion [LP67, BK62, Yaf92], on espère pouvoir montrer que la résolvante du Laplacien de M, $(-\Delta_g+\lambda^2)^{-1}$, a un prolongement *méromorphe* à un revêtement ramifié d'un ouvert de $\mathbb{C}$. Ce programme a été mené à bien dans le cas où les bouts sont convexes et hyperboliques réels [MM87], puis hyperboliques complexes [EMM91]. On peut aussi citer [GZ97, Gui05, GM12], qui complètent ces travaux. Ces démonstrations reposent sur un argument de *boîte noire* dont on trouve un exposé très clair dans [Bon01]

Depuis les travaux de Mazzeo et Melrose (voir [MM87]), on sait aussi traiter des cas *asymptotiquement* symétriques.

Les pôles de ce prolongement sont des objets intrinsèques de la métrique sur M, on les appelle *résonances*, et elles remplacent en quelque sorte les valeurs propres du cas compact.

Fonction d'Eisenstein et états résonnants.

Revenons au cas de $\mathbb{R}$, et perturbons le Laplacien par un potentiel V lisse, supporté dans $[-L, L]$. Considérons f_λ une fonction lisse sur $\mathbb{R}$ telle que

$$(-\Delta + V - \lambda^2)^k f_\lambda = 0 \text{ où } k \geqslant 1. \tag{$*$}$$

Alors en dehors du support de V, f_λ s'écrit forcément comme une somme $au_\lambda + bu_{-\lambda}$ où $u_\lambda(x) = e^{i\lambda x}$ et a, b sont des polynômes d'ordre $\leqslant 2k-2$. En suivant le vocabulaire introduit pour les états lagrangiens plus haut, on constate que u_λ correspond à une particule qui se propage vers la gauche si $\Re\lambda < 0$, et vers la droite si $\Re\lambda > 0$.

Si $\Re\lambda < 0$, u_λ est dit *entrant* dans $[L, +\infty[$, et *sortant* dans $]-\infty, -L]$. Ainsi, une solution de $(*)$ a une partie entrante et sortante, à gauche et à droite. Par ailleurs, si λ n'est pas réel, dans chaque demi-droite, un seul des deux entre u_λ et $u_{-\lambda}$ est dans L^2.

Étant donné que V est à support compact, il est assez facile de montrer que $-\Delta + V$ est encore auto-adjoint, de spectre continu $\mathbb{R}^+$. Éventuellement quelques valeurs propres discrètes peuvent apparaître, et on peut obtenir un prolongement méromorphe de la résolvante $(-\Delta + V - \lambda^2)^{-1}$, depuis $\{\Im\lambda > 0\}$. Grâce à cela, on constate que l'on peut construire des solutions de $(*)$ dont on a spécifié les parties non-L^2.

En particulier, on peut construire des solutions E_+, E_- pour $k = 1$ de $(*)$ qui vérifient pour $\lambda \in \mathbb{C}$

$$E_+(\lambda, x) = \mathbf{1}_{[L,+\infty[}(x)e^{-i\lambda x} + G_+(\lambda, x) \quad E_-(\lambda, x) = \mathbf{1}_{]-\infty,-L]}(x)e^{i\lambda x} + G_-(\lambda, x), \tag{3}$$

où $G_\pm$ sont dans L^2 pour $\Im\lambda > 0$. Les $E_\pm$ sont dites *fonctions d'Eisenstein.* Ce sont les solutions fondamentales de $(*)$, car toutes les autres peuvent être obtenues comme des combinaison linéaires de $\partial_\lambda^m E_\pm$. On se demande alors

> *Existe-t-il des solutions de* $(*)$ *qui soient purement entrantes, ou purement sortantes ?* De telles solutions sont appelées *états résonnants.*

Les fonctions $E_\pm$ sont en fait les fonctions propres généralisées, et en utilisant le théorème de Stone, on peut écrire le projecteur sur le spectre continu de $-\Delta + V$ comme

$$\Pi_c f = \frac{1}{4\pi}\sum_{\pm}\int_{\mathbb{R}} E(\lambda)\langle f, E(\lambda)\rangle d\lambda \text{ pour } f \in C_c^\infty(\mathbb{R}). \tag{4}$$

À partir de telles identités, on constate alors que les pôles de la résolvante prolongée sont exactement les pôles de la famille $E = (E_+, E_-)$. Revenons sur la forme que prend E_+ en dehors du support de V. On trouve que

$$E_{+|]-\infty,-L]} = \phi_{+-}(\lambda)e^{-i\lambda x} \quad E_{+|[L,+\infty[} = e^{-i\lambda x} + \phi_{++}(\lambda)e^{i\lambda x} \tag{5}$$

où ϕ_{+-} et ϕ_{++} sont méromorphes. Si on fait de même pour E_-, on obtient une matrice méromorphe 2×2. C'est la matrice qui dit ce qui se passe quand on envoit une onde plane sur le potentiel ; quelle partie de l'onde est diffusée, et dans quelle direction. Pour cette raison, cette matrice est appelée *matrice de diffusion*, et son déterminant $\varphi(\lambda)$, *déterminant de diffusion*.

En introduisant quelques outils théoriques, on arrive à faire un lien entre la résolvante et l'équation $(*)$. En effet, on trouve que les pôles de la résolvante sont exactement les pôles de φ, et qu'ils sont les valeurs $\lambda \in \mathbb{C}$, avec $\Im\lambda < 0$, pour lesquels il y a une solution de $(*)$ qui est soit purement entrante ($\Re\lambda < 0$), soit purement sortante ($\Re\lambda > 0$). De plus, si λ est un tel pôle, alors l'ensemble des solutions pour le paramètre λ constitue l'image de la partie polaire de la résolvante en λ.

D'un point de vue physique, les états résonnants sont les modes propres d'oscillation du système. Le fait qu'il aient une fréquence d'oscillation λ complexe correspond au fait qu'ils oscillent avec un amortissement exponentiel. Autrement dit, ils représentent des particules quantiques qui s'échappent de tout compact. On peut aussi les voir apparaître dans l'asymptotique pour l'équation des ondes en temps longs. Voir par exemple [NZ09, Vaĭ89, GN14, TZ00, CZ00].

Une telle description se retrouve dans certaines situations où on traite d'une perturbation à support compact du Laplacien d'une variété géométriquement finie dont les bouts sont exactement symétrique, même si elle peut devenir plus compliquée. Bony a montré que l'on pouvait même se passer de l'hypothèse que la perturbation est à support compact dans sa thèse [Bon01], dans certains cas.

Lien avec la dynamique

Formellement, comme le choix des mots le reflète, on peut voir $E_+(\lambda/h)$ comme un état lagrangien dont la phase (complexe) est $S = -\lambda x$. Cela correspond à une « particule » quantique micro-supportée sur la trajectoire (classique) du hamiltonien $H_V = |\xi|^2 + V(x)$ qui passe par $(x, -\Re\lambda)$ où $x >> L$. La partie complexe $\Im\lambda/h$ de la phase correspond à un amortissement le long de la trajectoire.

On espère donc faire le lien entre les propriétés de $E_\pm$ et la dynamique des trajectoires hamiltoniennes de H_V qui proviennent de l'infini. Pour suivre encore le vocabulaire précédent, ce sont des trajectoires qui en temps soit positif, soit négatif, s'échappent dans les bouts. Parmi ces trajectoires, certaines vont ensuite rester dans la partie compacte de la variété, d'autres vont repasser une infinité de fois par la partie compacte, et enfin les dernières vont s'échapper en temps positif *et* négatif dans les bouts. Dans les deux premiers cas, on dit que les trajectoires sont *captées*. Dans le dernier cas, on dit qu'elles sont *diffusées*.

On peut faire le même raisonnement sur une variété géométriquement finie générale. En fonction de la géométrie de la variété, les situations peuvent être très diverses.

1. D'une part, le système peut être très *ouvert*, auquel cas presque toutes les géodésiques sont diffusées. Dans ce cas, on va s'intéresser particulièrement à l'ensemble capté, i.e, l'ensemble des géodésiques qui ne s'échappe ni pour $t \to -\infty$, ni pour $t \to +\infty$. C'est en général un objet fractal, dont la dimension peut être reliée à la dimension de l'ensemble limite $\Lambda(\Gamma)$ défini plus haut.

Dans ce cas, comme il est facile de partir vers l'infini, on constate en général que l'énergie (dans l'équation des ondes, ou dans l'équation de Schrödinger, par exemple) s'échappe vers l'infini rapidement, et d'autant plus rapidement que la fréquence est élevée (ce qui correspond à une vitesse de propagation plus rapide). Dans le langage introduit plus haut, cela veut dire qu'il doit il y avoir peu de résonances de petite valeur imaginaire.

Les questions qui se posent dans de telles situations sont en général de déterminer exactement à quelle vitesse l'énergie s'échappe. Le domaine d'étude privilégié par la littérature est celui des variétés à bout euclidiens, ou des quotients convexe-cocompacts du plan hyperbolique réel, [Bor07, GN14, DG14, BH08]. Dans le cas de la courbure constante -1, c'est un problème fortement relié à l'étude de fonctions zêtas dynamiques [Nau05]

2. D'autre part, le système peut être presque fermé. Le cas compact correspond au cas complètement fermé, et une question par exemple est de comprendre les phénomènes qui se produisent quand on considère ces exemples qui ont en quelques sorte des *défaut de compacité* minimaux.

Des variétés à pointes

C'est dans ce contexte que se placent les travaux que j'ai réalisés dans ma thèse. L'exemple le plus simple de bout de volume fini est celui de la pointe hyperbolique. C'est topologiquement un cylindre

$$Z = [1, +\infty[_y \times \mathbb{T}^1_\theta \tag{6}$$

où $T^1 = \mathbb{R}/\mathbb{Z}$. Il est muni de la métrique

$$ds^2 = \frac{dy^2 + d\theta^2}{y^2}, \tag{7}$$

ce qui le rend isométrique à un quotient de l'horosphère $\{y > 1\}$ dans le demi-plan de poincaré $\mathbb{H}_2$ par la translation $z \to z + 1$. Il est donc de courbure constante -1, et on observe qu'il est de volume fini. Les surfaces dont les bouts sont des pointes hyperboliques sont les exemples les plus simples qui ont simultanément des bouts réguliers, ne sont pas compacts, mais tout de même de volume fini.

Les géodésiques dans une pointe qui ne partent pas verticalement finissent toujours par redescendre. Ainsi, très peu de trajectoires s'échappent (un ensemble de mesure nulle), et le flot géodésique est un système presque fermé. Remarquons ici que si on perturbe la métrique d'une pointe hyperbolique, il est possible de construire des exemples pour lesquels le comportement des trajectoires est assez différent (voir [DOP00] par exemple). On se restreint donc au cas des pointes exactement hyperboliques.

Par ailleurs, presque tous les résultats que j'obtiens ne sont pas plus difficiles quand on remplace une pointe de dimension 2 par des pointes de dimension supérieures, qui sont des quotients de $\mathbb{H}^{d+1}$ par des sous groupe discrets paraboliques maximaux. Je donne donc les résultats en toutes dimensions quand c'est possible.

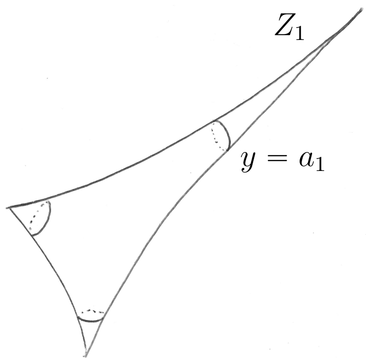


FIGURE 1 : La courbe modulaire $PSL_2(\mathbb{Z})\backslash\mathbb{H}^2$.

Avant de décrire la théorie spectrale de telles surfaces avec plus de détails, ce qui sera l'objet de la prochaine partie, voyons comment elles ont pu apparaître dans un autre domaine des mathématiques.

La courbe modulaire et Selberg

La première surface de ce type a avoir été étudiée est probablement la courbe modulaire $PSL_2(\mathbb{Z})\backslash\mathbb{H}_2$. Techniquement, c'est un orbifold, avec une pointe, et deux singularités coniques. Elle a un revêtement d'ordre 6 qui est une surface proprement dite, avec 3 pointes. Plus généralement, si $\Gamma(n)$ est le noyau de la réduction modulo n, $\pi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_n)$, on peut considérer la surface $\Gamma(n)\backslash\mathbb{H}_2$. C'est une surface de Riemann et une surface à pointe.

De nombreux théorèmes d'arithmétique peuvent être traduits en propriétés spectrales et dynamiques pour ces surfaces. Du côté dynamique, on peut observer que les géodésiques diffusées (i.e, les géodésiques qui partent à l'infini en temps positif *et* négatif) sur la courbe modulaire correspondent aux géodésiques verticales dans le demi-plan de Poincaré dont l'abscisse est un nombre rationnel. Cela conduit à Belabas, Hersonski, Paulin à proposer de les appeler géodésiques *rationnelles*, [BHP01]

Du côté spectral, Selberg [Sel89a, p. 670] a montré que le déterminant de scattering φ pour la courbe modulaire s'écrit

$$\varphi(s) = \sqrt{s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}, \tag{8}$$

où ζ est la fonction zéta de Riemann, et Γ la fonction Gamma d'Euler. Par exemple, les résonances sont situées sur la droite $\{\Re s = 1/4\}$ si et seulement si l'hypothèse de Riemann est satisfaite.

Selberg a effectué des travaux pionniers dans cette direction. L'exemple le plus connu est peut-être sa formule de Trace. Elle donne un lien entre les longueurs des géodésiques fermées de la surface et les poles de φ, dans le cas où la surface M est une surface à pointe de courbure constante -1 .

Tachons d'en donner un énoncé précis. Soit h une fonction holomorphe sur la bande $\{|\Im r| \leqslant 1/2 + \epsilon\}$, paire, et satisfaisant $|h(s)| = \mathcal{O}(1/|s|^{2+\epsilon})$, où $\epsilon > 0$. Alors si les $1/4 + r_i^2$

sont les valeurs propres du Laplacien sur M, avec $\Re r_i \geqslant 0$, on trouve

$$\begin{aligned}\sum h(r_i) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)h(r)dr \\ = vol(M)\int_{\mathbb{R}} r\tanh(\pi r)h(r)dr - \frac{\kappa}{\pi}\int_{\mathbb{R}}\frac{\Gamma'}{\Gamma}\left(1+ir\right)h(r)dr \\ - 2\kappa\log 2g(0) + \frac{1}{2}(\kappa - \Re\varphi(1/2))h(0) \\ + 2\sum_{\{\gamma\}}\sum_{k=1}^{\infty}\frac{\log\ell(\gamma)}{\ell(\gamma)^{1/2}-\ell(\gamma)^{-1/2}}g(k\log\ell(\gamma)).\end{aligned}$$

(Formule de Trace de Selberg)

Dans la somme, $\{\gamma\}$ parcours la liste des *géodésiques fermées primitives*, $\ell(\gamma)$ étant la longueur d'une telle géodésique, $g = \hat{h}$, et κ est le nombre de pointes. Attention, dans le cas d'un orbifold (par exemple, la courbe modulaire), il faut rajouter des termes liés aux classes de conjugaisons d'éléments elliptiques. On peut trouver la preuve de Selberg dans [Sel89a, Harmonic Analysis]. De cette égalité, Selberg a aussi déduit [Sel89b, (0.2)] cette loi de Weyl :

$$\begin{aligned}\#\{0 \leqslant r_i \leqslant T\} - \frac{1}{2\pi}\int_0^T \frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)dr \\ = \frac{vol(M)}{4\pi}T^2 - \frac{\kappa}{\pi}T\log t + \frac{\kappa(1-\log 2)}{\pi}T + \mathcal{O}\left(\frac{T}{\log T}\right).\end{aligned}$$

(Loi de Weyl pour les surface à pointes hyperbolique)

Après Selberg, de nombreux mathématiciens ont mis à profit cette correspondance. Voir [Hej76, Iwa02, Ber11]. On peut aussi consulter [Sar90].

Chapitre 1

Introduction à la théorie spectrale du Laplacien sur les variétés à pointes

Comme on a pu le voir dans les lignes précédentes, la théorie spectrale du Laplacien sur les surfaces à pointes n'est pas un sujet d'étude récent. Après avoir énoncé quelques définitions et présenté des résultats fondamentaux, il convient de dresser un panorama des résultats plus avancés contenus dans la littérature. Ensuite, je détaillerai les contributions à la théorie que l'on trouvera dans cette thèse.

1.1. Propriétés élémentaires

1.1.1. Quelques notations

Comme nous l'avons dit, une variété à pointe est la réunion disjointe d'une partie compacte à bords et d'un nombre fini de pointes exactement hyperboliques :

$$M = M_0 \sqcup Z_1 \sqcup \cdots \sqcup Z_\kappa. \tag{1.1}$$

Le nombre de pointes est $\kappa \geqslant 1$. Chaque pointe est isométrique à :

$$Z_i \simeq]a_i, +\infty[_y \times \left\{\mathbb{R}^d/\Lambda_i\right\}_\theta . \tag{1.2}$$

où Λ_i est un réseau de covolume 1 dans $\mathbb{R}^d$ et $a_i > 0$ est un nombre réel. La métrique est donnée par

$$ds^2_{|Z_i} = \frac{dy^2 + d\theta^2}{y^2} \tag{1.3}$$

où $d\theta^2$ est la métrique canonique sur $\mathbb{R}^d$. On peut calculer que $\mathrm{vol}(Z_i) = (a_i d)^{-1} < \infty$. Dans tout ce qui suit, K sera la courbure sectionnelle de la variété.

Remarque 1.1. *La condition que* Λ_i *est de covolume* 1 *est une condition de* normalisation. *On aurait aussi pu demander que les pointes s'arrêtent à la hauteur* $y = 1$, *auquel cas il faudrait remplacer* Λ_i *par* $a_i\Lambda_i$.

Si la pointe dont il est question est claire, on fera référence à la coordonnée y dans la carte (1.2) comme la *hauteur* y, sans autre forme de procès. Si nécessaire, on écrira y_i pour préciser la pointe. Enfin, y_M sera une fonction lisse strictement positive sur M correspondant avec y dans Z_i, et plus petite que $a_0 := \max a_i$ dans M_0. Les tranches $\{y =$ constante $\}$ seront appelées *horosphères projetées*, ou plus informellement *horosphères*.

Les fonctions d'Eisenstein

Comme annoncé, les *fonctions d'Eisenstein* sur une variété à pointe forment l'unique famille méromorphe $E(s) = (E_i(s))_{i=1\dots\kappa}$ de fonctions sur la variété qui vérifient les propriétés suivantes. Tout d'abord,

$$(-\Delta - s(d-s))E(s) = 0. \tag{1.4}$$

Ensuite, si χ_i est une fonction supportée dans la pointe Z_i, qui vaut 1 pour y assez grand, alors $E_i - \chi_i y^s$ est une fonction L^2 pour $\Re s > d/2$ et $s \notin [d/2, d]$. Dans ce qui suit, le vecteur de fonctions $E(s)$ sera compris comme un vecteur colonne.

L'unicité d'une telle famille est relativement évidente, et l'existence pour les paramètres s tels que $\Re s > d/2$ est facile. Il suffit de considérer les fonctions

$$\chi_i y^s + (-\Delta - s(d-s))^{-1}[\Delta, \chi_i] y^s. \tag{1.5}$$

Pour montrer que cette famille se prolonge de façon méromorphe, l'argument remonte à Colin de Verdière, et consiste à utiliser les fameux *Pseudo-Laplaciens*. Il permet aussi de montrer que la résolvante

$$R(s) = (-\Delta - s(d-s))^{-1}, \tag{1.6}$$

se prolonge depuis $\Re s > d/2$ comme une famille méromorphe sur $\mathbb{C}$ d'opérateurs $C_c^\infty \to C^\infty$.

Nous n'allons pas reproduire toute la preuve ici, que l'on peut trouver dans [CdV83, Mül92, Mül83]. Néanmoins, il est éclairant d'en rappeler quelques idées.

Le Pseudo-laplacien

Soit $\mathcal{H}_a$ l'espace des fonctions L^2 sur M telles que leur valeur moyenne sur les horosphères projetées (mode nul de Fourier en θ) de hauteur $y \geqslant a \geqslant a_0$ s'annule. La restriction de la norme H^1 à $\mathcal{H}_a$ permet de définir un opérateur Δ_a à résolvante compacte. Colin de Verdière a introduit cet opérateurs sous le nom de pseudo-laplacien. Il coincide bien sûr avec le laplacien pour les fonctions qui sont dans $\mathcal{H}_{a'}$ pour $a_0 \leqslant a' < a$. Le prolongement de la résolvante de Δ à partir de Δ_a est un cas d'école de la méthode de Boite noire généralisée par Sjostrand et Zworki [SZ91, Bon01].

Le laplacien dans les pointes préserve les modes de Fourier en θ. Dans les modes $k \neq 0$, on peut calculer qu'il est conjugué à des opérateurs du type

$$-\frac{\partial^2}{\partial r^2} + e^{2r}, \text{ sur } L^2(]r_k, +\infty[, dr). \tag{1.7}$$

C'est un opérateur de Schrödinger avec un potentiel fortement confinant, dont la résolvante est bien sûr compacte. Le spectre continu est donc produit seulement par le mode nul en Fourier, autrement dit les fonctions qui ne dépendent pas de θ. C'est exactement ce fait qui est derrière la méthode des Pseudo-laplaciens de Colin de Verdière.

On notera toujours dans la suite $a_0 = \max a_i$ et

$$\Pi_y^* \text{ le projecteur sur } \mathcal{H}_y, \text{ pour } y \geqslant a_0. \tag{1.8}$$

La matrice de diffusion

On s'intéresse donc au mode nul de Fourier des fonctions d'Eisenstein. On observe qu'ils s'écrivent nécessairement comme :

$$\int E_{i|Z_j} d\theta_j = \delta_{ij} y^s + \phi_{ij}(s) y^{d-s}. \tag{1.9}$$

Le coefficient $\phi_{ij}(s)$ dépend méromorphiquement de s et on appelle $\phi(s) = (\phi_{ij}(s))$ la *matrice de diffusion*. On note aussi $\varphi(s)$ le déterminant de la matrice de diffusion ; il est logiquement appelé *déterminant de diffusion*. D'après la propriété d'unicité, on trouve que

$$E(s) = \phi(s)E(d-s) \text{ et } E^T(s) = E^T(d-s)\phi(s). \tag{1.10}$$

Dans cette équation, M^T est la transposée de la matrice M. Il vient que

$$\phi(s)^T = \phi(s) \quad \phi(s)\phi(d-s) = \mathbb{1} \quad \varphi(s)\varphi(d-s) = 1 \tag{1.11}$$

De plus, $E(s)$ est réel pour s réel, donc $\overline{E(s)} = E(\overline{s})$ et $\phi(\overline{s}) = \overline{\phi(s)}$, et on obtient la relation principale

$$\overline{\phi}^T \phi\left(\frac{d}{2} + it\right) = \mathbb{1}. \tag{1.12}$$

En particulier, le déterminant de diffusion est de module 1 sur la droite $\{Res = d/2\}$, appelée *axe unitaire*. On peut donc définir

$$\mathcal{S}(T) = \frac{1}{2i\pi} \log \varphi\left(\frac{d}{2} + iT\right). \tag{1.13}$$

Autrement dit

$$\mathcal{S}(T) = \frac{1}{4\pi} \int_{-T}^{T} \frac{\varphi'}{\varphi}\left(\frac{d}{2} + it\right) dt. \tag{1.14}$$

Spectre et décomposition spectrale

On peut écrire la décomposition spectrale du laplacien sous la forme suivante [Mül83]

1. Il y a du spectre discret $\sigma_d = \{\lambda_0 = 0 < \lambda_1 \leqslant \dots \leqslant \lambda_i \leqslant \dots\}$, éventuellement fini. Certaines valeurs propres peuvent être plongées dans le spectre continu. À chaque $\lambda \in \sigma_d$, on associe une base orthonormée u_λ^i de fonctions propres pour la valeur propre λ. De plus, on note $\lambda_i = d^2/4 + r_i^2 = (d/2 + ir_i)(d - (d/2 + ir_i))$ avec $\Re r_i \geqslant 0$ et $\Im r_i \leqslant 0$. Quand on sommera sur les r_i, on les comptera avec la multiplicité de la valeur propre λ_i.

2. Il y a aussi du spectre absolument continu $\sigma_{ac} = [d^2/4, +\infty[$, dont la multiplicité est κ, le nombre de pointes.

Pour $f \in C_c^\infty(M)$, on a

$$f = \sum_\lambda \langle f, u_\lambda \rangle u_\lambda + \frac{1}{4\pi} \int_{\mathbb{R}} \sum_i \langle f, E_i \rangle E_i \left(\frac{d}{2} + it\right) dt, \tag{1.15}$$

ce qui est un raffinement de la décomposition spectrale du laplacien. La preuve de ces résultats peut être trouvée dans [Mül83].

Étant donné la façon dont les fonctions d'Eisenstein, et le déterminant de diffusion, ont été construits, on obtient le fait suivant

Proposition 1.1.1. *Les pôles du prolongement méromorphe de la résolvante $R(s)$ sont*

$$\left\{\frac{d}{2} \pm ir_j,\ j\right\} \cup \{ \text{ pôles de } \varphi \ \}. \tag{1.16}$$

De plus, les pôles de φ sont aussi exactement les pôles de $E(s)$ et de $\phi(s)$, de même multiplicités.

Suivant Müller, on appelle *ensemble résonnant* l'ensemble $\mathrm{Res}(M, g)$ des pôles de $R(s)$, et *résonances* l'ensemble $\mathcal{R}$ des pôles de $\varphi(s)$. On compte leur multiplicité comme pôles de $R(s)$. Les résonances seront l'objet central des travaux qui suivent.

Étant donné que $\varphi(s)\varphi(d-s) = 1$, si ρ est un pôle de φ dans $\{\Re s < d/2\}$, $d - \rho$ est un zéro de φ dans $\{\Re s > d/2\}$. En pratique, il sera souvent plus facile de travailler avec les zéros de φ plutôt qu'avec ses pôles.

Remarquons aussi la chose suivante. Les fonctions propres L^2 pour une valeur propre plongée plus grande que $d^2/4$ ont nécessairement un mode nul de Fourier identiquement nul. Par contre pour les petites valeurs propres dans $[0, d^2/4[$, ce n'est pas nécessairement le cas. Ceci fait que φ peut avoir des pôles dans $]d/2, d]$, qui correspondent aux éventuels r_j imaginaires pures. Dans ce cas, φ a des zéros dans $[0, d/2[$.

1.1.2. 0-intégrales et Maass-Selberg

Relations de Maass-Selberg

Il est utile de définir, pour $y > a_0 = \max a_i$ les fonctions suivantes

$$G_i^y(s) := \Pi_y^* E_i(s). \tag{1.17}$$

Tant que s n'est pas un pôle de E, les G_i^y sont dans L^2. On forme la matrice $V(s)$ dont les coefficients sont

$$V_{ij}(s) = \int_M G_i^y(s)\overline{G_j^y(s)}. \tag{1.18}$$

On note aussi $f_{ij} = \delta_{ij} y^s + \phi_{ij}(s) y^{d-s}$ le mode nul de Fourier de E_i dans la pointe Z_j. En utilisant le théorème de Stokes, on peut montrer que

$$(d-2s)\int G_i^y(s)\overline{G_j^y(s)} + 2i\Im(s(d-s))\int \partial_s G_i^y \overline{G_j^y(s)} = y^{1-d}\sum_k \left[\partial_s f_{ik}\overline{\partial_y f_{jk}} - \overline{f_{jk}}\partial_y\partial_s f_{ik}\right]. \tag{1.19}$$

où la somme à la fin est entendue prise au point y. Pour $\Re s > d/2$, cela donne

$$V(s) = \frac{y^{2\Re s - d} - y^{d-2\Re s}\phi\phi^*}{2\Re s - d} + \frac{\phi^* y^{2i\Im s} - \phi y^{-2i\Im s}}{2i\Im s} \quad \text{quand } \Re s > d/2. \tag{MS-1}$$

En prenant la limite pour $\Re s = d/2$, on a aussi

$$V\left(\frac{d}{2} + it\right) = 2\log y \mathbb{1} - \phi'\phi^* + \frac{y^{2it}\phi^* - y^{-2it}\phi}{2it}. \tag{MS-2}$$

Enfin, en prenant la trace de cette matrice, on obtient

$$\left\| G^y\left(\frac{d}{2} + it\right)\right\|^2 = 2\kappa \log y - \frac{\varphi'}{\varphi}\left(\frac{d}{2} + it\right) + \mathrm{Tr}\,\frac{y^{2it}\phi^* - y^{-2it}\phi}{2it}. \tag{MS-3}$$

Ces relations sont des cas particuliers du lemme 7.23 dans [Mül83].

0-intégrales

Nous allons utiliser tout de suite les relations précédentes, mais d'abord, nous avons besoin d'une définition.

Definition-Proposition 1.1.2. *Soit f une fonction continue sur $\mathbb{R}^+$. On suppose que*

$$\int_0^{1/\epsilon} f(r)dr = R\left(\frac{1}{\epsilon}, \log \epsilon\right) + o_{\epsilon\to 0}(1), \tag{1.20}$$

où R est une fraction rationnelle. Alors si c est la constante dans le développement de Laurent en $(+\infty, +\infty)$, on pose

$$^0\!\!\int_0^{+\infty} f = c, \tag{1.21}$$

et on appelle ceci la 0-intégrale de f. C'est une façon de renormaliser l'intégrale de f.

Maintenant, si A est un opérateur $C_c^\infty(M) \to (C_c^\infty(M))'$, de noyau de Schwarz $K(x, x')$ par rapport au volume de M, et si A est de classe trace, on trouve que

$$\operatorname{Tr} A = \int_M K(x, x) d\operatorname{vol}(x). \tag{1.22}$$

Supposons maintenant que A n'est pas forcément de classe trace, mais que comme la définition 1.1.2,

$$\int_{M, y\leqslant 1/\epsilon} K(x, x) d\operatorname{vol} = R\left(\frac{1}{\epsilon}, \log\epsilon\right) + o(1).$$

Alors on prend encore c la partie constante du développement de R, et on pose

$$0\text{-}\operatorname{Tr} A := {}^0\!\!\int_M K(x, x) dvol(x) = c. \tag{1.23}$$

C'est la 0-trace de A, introduite par Selberg. On trouve dans [GZ97] un exposé plus détaillé de cet objet.

Traces des fonctions du Laplacien

La 0-trace présentée précédemment est surtout utile pour les fonctions du Laplacien. En effet dans les surfaces à pointes, étant donné que le Laplacien n'est pas à résolvante compacte, aucune fonction du Laplacien n'est à trace. Par contre, on peut définir des 0-traces pour certaines fonctions du Laplacien. En effet, d'après (MS-3) on trouve que si f est à décroissance rapide, et C^1 autour de $d^2/4$,

$$0\text{-}\operatorname{Tr} f(-\Delta) = \sum_i f\left(\frac{d^2}{4} + r_i^2\right) - \frac{1}{4\pi}\int_{\mathbb{R}} f\left(\frac{d^2}{4} + t^2\right) \frac{\varphi'}{\varphi}\left(\frac{d}{2} + it\right) dt + \frac{1}{4} f(d^2/4) \operatorname{Tr}\phi(d/2). \tag{1.24}$$

Formellement, on en déduit en prenant $f(x) = \delta(\sqrt{x - d^2/4} - t)$.

$$^0\!\!\int \left| E\left(\frac{d}{2} + it\right)\right|^2 d\operatorname{vol} = -\frac{\varphi'}{\varphi}\left(\frac{d}{2} + it\right) + \delta(t)\frac{1}{4}\operatorname{Tr}\phi(d/2). \tag{1.25}$$

1.1.3. Comptage et Loi de Weyl

Loi de Weyl pour la densité spectrale

Il existe un certain nombre de résultats de comptage et d'estimées de Weyl, nous allons essayer de leur rendre justice. Tout d'abord, il convient d'insister sur le fait que l'objet qui remplace la fonction de comptage du cas compact est la 0-trace du projecteur spectral :

$$N(T) = 0\text{-Tr}\left\{\mathbf{1}_{[0,d^2/4+T^2]}(-\Delta)\right\} = N_d(T) + \mathcal{S}(T) + \operatorname{Tr}\phi(d/2)/4 \tag{1.26}$$

où

$$N_d(T) = \#\{\lambda \text{ valeur propre de } -\Delta \text{ telle que } \lambda \leqslant d^2/4 + T^2\}. \tag{1.27}$$

et $\mathcal{S}$ est la phase de diffusion introduite en (1.13). Notons que le membre de droite de (1.26) contient toutes les informations spectrales. D'après Selberg, pour une surface *hyperbolique* à pointe, on a

$$N(T) = \frac{vol(M)}{4\pi}T^2 - \frac{\kappa}{\pi}T\log T + \frac{\kappa(1-\log 2)}{\pi}T + \mathcal{O}\left(\frac{T}{\log T}\right). \tag{WS}$$

Le nombre κ est encore le nombre de pointes, et le terme $T\log T$ est en quelque sorte une contribution des pointes. Müller a ensuite montré [Mül92] dans le cas des surfaces avec de la courbure variable dans un compact, que

$$N(T) = \frac{vol(M)}{4\pi}T^2 + o(T^2) \tag{WM1}$$

Dans [Mül86], Il a en fait donné une preuve d'une version plus générale de ce résultat, qui fonctionne en toute dimension, avec une définition plus générale des pointes. Il autorise les tranches à être des variétés riemanniennes compactes générale, plutôt que seulement des tores. Cela donne (avec le même préfacteur que dans le cas compact)

$$N(T) = \frac{vol(M)}{(4\pi)^{(d+1)/2}\Gamma(d/2+3/2)}T^{d+1} + o(T^{d+1}). \tag{WM2}$$

Dans un article ultérieur, Parnovski [Par95] a montré que dans le cas des surfaces, on peut améliorer le reste dans (WM1) et

$$N(T) = \frac{vol(M)}{4\pi}T^2 - \frac{\kappa}{\pi}T\log T + \mathcal{O}(T) \tag{WP1}$$

en général. Quand l'ensemble des géodésiques périodiques est de mesure nulle, il obtient

$$\frac{vol(M)}{4\pi}T^2 - \frac{\kappa}{\pi}T\log T + \frac{\kappa(1-\log 2)}{\pi}T + o(T). \tag{WP2}$$

Cette condition sur les géodésiques périodiques est classique depuis les travaux de Duistermaat-Guillemin [DG75].

Loi de Weyl pour les résonances.

Dans le cas des variétés compactes, la loi de Weyl compte effectivement des états propres discrets. Il est assez simple de se demander si on peut traduire les estimées (WM2) en des estimées de comptage pour les résonances. Il n'est pas forcément aisé d'y répondre.

Dans le cas des surfaces hyperboliques, Selberg a montré que les résonances sont contenues dans une bande verticale $\{d/2 > \Re s > -\delta\}$. Nous aurons l'occasion de revenir sur ce résultat. Quoi qu'il en soit, il obtient que si $N_{\mathcal{R},\delta}(T)$ est le nombre de résonances dans cette bande, de partie imaginaire comprise entre 0 et T,

$$N_d(T) + N_{\mathcal{R},\delta}(T) = \frac{vol(M)}{4\pi}T^2 - \frac{\kappa}{\pi}T\log T + \frac{\kappa(1-\log 2) - \mathcal{T}^0/2}{\pi}T + \mathcal{O}\left(\frac{T}{\log T}\right) \quad \text{(WSR)}$$

où $\mathcal{T}^0$ est une constante géométrique (un plus petit temps de séjour, voir chapitres 3 et 4.2).

Dans le cas de la courbure variable, il n'est pas vrai en général que les résonances sont contenues dans une bande. On introduit donc la fonction de comptage $N_{\mathcal{R}}(T)$ qui compte le nombre de résonances dont le module est plus petit que T. Müller a montré que

$$N_d(T) + \frac{1}{2}N_{\mathcal{R}}(T) = \frac{vol(M)}{4\pi}T^2 + o(T^2) \quad \text{(WMR)}$$

Toujours dans [Par95], Parnovski a aussi montré pour tout $\epsilon > 0$,

$$N_d(T) + \frac{1}{2}N_{\mathcal{R}}(T) = \frac{vol(M)}{4\pi}T^2 + o(T^{3/2+\epsilon}) \quad \text{(WPR1)}$$

Dans [Bon14b], en utilisant des arguments élémentaires, un résultat de [PZ99] et les résultats de [Par95], j'ai montré qu'en fait

$$N_d(T) + \frac{1}{2}N_{\mathcal{R}}(T) = \frac{vol(M)}{4\pi}T^2 + \mathcal{O}(T^{3/2}). \quad \text{(WBR)}$$

Enfin, Parnovski montrait aussi que si l'on suppose en courbure variable que les résonances sont tout de même contenues dans une bande $\{d/2 > \Re s > -\delta\}$, alors pour tout $\epsilon > 0$,

$$N_d(T) + \frac{1}{2}N_{\mathcal{R}}(T) = \frac{vol(M)}{4\pi}T^2 + \mathcal{O}(T^{1+\epsilon}) \quad \text{(WPR2)}$$

Résultats de factorisation pour φ

Dans son article [Mül92], Müller a donné une factorisation pour le déterminant de diffusion dans le cas des surfaces ($d = 1$, et courbure variable dans un compact), sous la forme (c'est son Théorème 3.31)

$$\varphi(s) = \varphi\left(\frac{1}{2}\right) q^{s-1/2} \prod_{\rho} \frac{s-1+\overline{\rho}}{s-\rho}. \quad (1.28)$$

où q est une constante, et où ρ parcourt l'ensemble des pôles de φ, i.e l'ensemble des résonances $\mathcal{R}$. En passant à la dérivée logarithmique on trouve pour $t \in \mathbb{R}$

$$\frac{\varphi'}{\varphi}\left(\frac{1}{2} + it\right) = \log q + \sum_{\rho\in\mathcal{R}} \frac{2\Re\rho - 1}{(1/2 - \Re\rho)^2 + (t - \Im\rho)^2} \quad (1.29)$$

Au cours de la preuve de la convergence du produit infini (1.28), il montre aussi

$$\sum_{\rho} \frac{2\Re\rho - 1}{|\rho - 1/2|^2} < \infty. \tag{1.30}$$

En fait

Proposition 1.1.3. *Les assertions* (1.28), (1.29) *et* (1.30) *sont encore valable en dimension plus grandes, en remplaçant* 1/2 *par* $d/2$ *:*

$$\varphi(s) = \varphi\left(\frac{d}{2}\right) e^{sQ(s^2)} \prod_{\rho\in\mathcal{R}} \frac{s - d + \overline{\rho}}{s - \rho} \tag{1.31}$$

$$\frac{\varphi'}{\varphi}\left(\frac{d}{2} + it\right) = Q(s^2) + 2s^2Q'(s^2) + \sum_{\rho\in\mathcal{R}} \frac{2\Re\rho - d}{(d/2 - \Re\rho)^2 + (t - \Im\rho)^2}; \tag{1.32}$$

$$\sum_{\rho\in\mathcal{R}} \frac{2\Re\rho - d}{|\rho - d/2|^2} < \infty. \tag{1.33}$$

Dans (1.31), Q *est un polynôme de degré au plus* $\lfloor d/2 \rfloor$.

Je ne connais pas de preuve dans la littérature de ces faits, en voici donc une.

Démonstration. Il s'agit de remarquer que la preuve donnée par Müller s'étend facilement en toutes dimensions. C'est le principal résultat de la section 3 de l'article de Müller [Mül92], des pages 274 à 280. Il y a plusieurs étapes

1. D'abord, il faut montrer que

$$\varphi = \frac{F_1}{F_2} \tag{1.34}$$

où F_1 et F_2 sont des fonctions entières d'ordre 2. Dans notre cas, elles seront d'ordre $d+1$, et le même schéma de preuve fonctionne. Le seul ingrédient qui n'est pas déjà dans [Mül92] est le fait suivant. On a besoin de savoir que si $N_a(T)$ est la fonction de comptage des valeurs propres du pseudo-laplacien Δ_a, alors

$$N_a(T) \leqslant CT^{d+1}. \tag{1.35}$$

Mais ce résultat est déjà présent dans [Mül86], c'est la conséquence des équations (4.10), (4.11) et (4.12) de cet article.

2. Il s'agit ensuite de montrer la convergence de la somme (1.33). Pour ce faire, on utilise le théorème de Carleman [Tit58, Theorem 3.71]. Pour pouvoir s'en servir, il faut trouver une fonction bornée sur le demi-plan $\{\Re s > d/2\}$ dont les zéros sont exactement les zéros de φ.

Si on arrive à trouver $q \geqslant 1$ tel que $|\varphi(s)| \leqslant Cq^{\Re s}$ pour $\Re s > d/2 + 2$ comme dans le lemme 3.21 de [Mül92], on va considérer $\xi(s)$ comme dans l'équation (3.26) de Müller :

$$\xi(s - d/2) = q^{d/2-s} \prod_{\rho\in\mathcal{R}, \Re\rho\geqslant d/2} \frac{s - \rho}{s - d + \overline{\rho}} \varphi(s) \tag{1.36}$$

Les pôles de φ dans $\Re s > d/2$ sont en nombre fini, et la fonction ξ ainsi obtenue est de module 1 sur $\Re s = 0$, bornée dans $\{\Re s > 2\}$.

Pour s'assurer que $\xi(s)$ est bornée sur le demi-plan $\{\Re s > 0\}$, il suffit d'utiliser (MS-1) comme dans la section 4.2.2. En effet, cela implique que pour $a_0 = \max a_i$ comme précédemment,

$$\phi\phi^* \leqslant a_0^{2(2\Re s-d)} + \frac{2\Re s-d}{\Im s}\frac{a_0^{2s}\phi^* - a_0^{2\overline{s}}\phi}{2ia_0^d}. \tag{1.37}$$

en tant que matrice hermitiennes. Mais on sait que $a_0^{2s}\phi^* - a_0^{2\overline{s}}\phi/2i \leqslant a_0^{2\Re s}\sqrt{\phi\phi^*}$, aussi en tant que matrices hermitiennes. En considérant $u \in \mathbb{C}^\kappa$ avec $\|u\| = 1$, on montre que

$$\langle\phi\phi^* u, u\rangle \leqslant (a_0^{2\Re s-d})^2\left(1 + \frac{2\Re s-d}{\Im s}\sqrt{\langle\phi\phi^* u, u\rangle}\right). \tag{1.38}$$

On peut conclure

$$|\varphi(s)| \leqslant a_0^{\kappa(2\Re s-d)}\left\{\sqrt{1+\frac{|\Re s-d/2|^2}{|\Im s|^2}} + \frac{|\Re s-d/2|}{|\Im s|}\right\}^\kappa. \tag{1.39}$$

Ce schéma de preuve est dû à Selberg.

3. Maintenant que l'on sait que ξ est bornée sur le demi-plan, on obtient la convergence de la somme (1.33). On en déduit la convergence du produit dans le membre de droite de (1.31). Ceci implique que

$$\varphi(s) = e^{\tilde{Q}(s)}\prod_{\rho\in\mathcal{R}}\frac{s-d+\overline{\rho}}{s-\rho} \tag{1.40}$$

où $\tilde{Q}$ est un polynôme de degré au plus $d+1$. Les conditions de symmétrie sur φ impliquent que $\tilde{Q} = il\pi + P$ où $l \in \mathbb{Z}$, et P est un polynôme réel dont les coefficients pairs sont nuls. Autrement dit

$$\tilde{Q}(s) = il\pi + a_1 s + a_3 s^3 + \cdots + a_{2[d/2]+1}s^{2[d/2]+1}. \tag{1.41}$$

Ceci conclut la preuve. □

Une formule de Trace

Müller a aussi montré la formule suivante pour les surfaces [Mül92]. Si $g \in \mathscr{S}(\mathbb{R})$ est paire, et $h = \hat{g}$, on note $h_+(z) = \int_0^{+\infty} g(y)e^{zy}dy$ pour $\Re z \leqslant 0$. Alors

$$\begin{aligned}-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)h(r)dr \\ = -\frac{\log q}{4\pi}g(0) + \frac{1}{2}\sum_\rho n(\rho)\left\{h_+(\rho-1/2) + h_+(\overline{\rho}-1/2)\right\}\end{aligned} \tag{1.42}$$

où ρ parcourt les pôles *et* les zéros de φ dans $\{\Re z < 1/2\}$. L'entier $n(\rho)$ est la multiplicité comme pôle de φ (les zéros ont une multiplicité négative !).

On peut probablement étendre cette formule en toute dimension, mais nous n'en aurons pas besoin.

1.2. Résultats plus fins

1.2.1. La stabilité des valeurs propres discrètes

Dans le cas de la courbe modulaire, le spectre discret est infini, et Selberg avait remarqué que le terme correspondant au spectre discret dans la formule de Weyl (1.26) était bien plus grand que celui associé au spectre continu. Il avait conjecturé que ce phénomène serait encore vrai dans le cas d'une surface à pointe hyperbolique générale.

En 1981, dans [CdV81], Y. Colin de Verdière montrait le prolongement de la résolvante pour les surfaces à pointes de courbure variable dans un compact. Par la suite, dans [CdV83], il montrait que parmi les perturbations à support compact d'une métrique de courbure constante sur une surface à pointe, la plupart des métriques obtenues n'ont qu'un nombre fini de valeur propres discrètes, et qu'elles sont dans le segment $[0, 1/4]$. L'idée de la preuve consistait à montrer que lors d'une pertubation, les valeurs propres discrètes contenues dans le spectre continu étaient instables, et devenaient des résonances. Ceci montre que l'idée de Selberg n'était pas valide dans le cas (plus général) des surfaces à pointes de courbure *variable*. Müller a étendu la preuve de Colin de Verdière en toute dimension dans [Mül86].

Par la suite, Phillips et Sarnak [PS94] ont montré qu'en fait, une valeur propre discrète qui correspond à $1/2 + ir$ sur l'axe unitaire, est instable dès qu'une certaine série de Dirichlet construite à partir de la métrique ne s'annule pas au point $1/2 + ir$. Ce genre de résultat est reliée à des résultats de physique, et on s'y réfère comme *Fermi's Golden Rule* ; voir [DZot, section 4.4.2]. Cela semble indiquer que cette conjecture de Selberg est fausse, et que génériquement, même parmi les surfaces à pointes *hyperboliques*, il n'y a qu'un nombre fini de valeur propres discrètes. Ceci a donné lieu à de nombreux travaux [JP97, PR13].

Comme nous serons principalement intéressés par les exemples de courbure variable, nous allons nous concentrer sur les séries d'Eisenstein, et sur le spectre continu. Les valeurs propres discrètes et les fonctions propres associées ne recevront que peu d'attention.

1.2.2. Métrique d'ensemble résonnant fixé

L'étude des valeurs propres d'une variété riemannienne compacte est souvent motivée par la question suivante :

Peut-on entendre la forme d'un tambour ?

En effet, les fréquences d'oscillation de la peau d'un tambour peuvent (au moins en première approximation) être décrites comme les valeurs propres du Laplacien sur la surface du tambour, avec des condition de Dirichlet au bord. La non-moins célèbre réponse est

Non !

En effet, on peut construire des exemples de variétés (M, g), (M', g') non isométriques, qui ont le même spectre du laplacien (on les appelles *isospectrales*). Le premier exemple est dû à Milnor [Mil64], ce sont deux tores de dimension 16. C'est Sunada [Sun85] qui a construit une méthode pour construire des familles d'exemples. Par ailleurs, Mc Kean

[McK72] et Wolpert [Wol79] ont montré qu'il ne peut y avoir au plus qu'un nombre fini de surfaces hyperboliques compactes non isométriques avec les mêmes valeurs propres.

Dans le cas des surfaces à pointes *hyperboliques*, des résultats similaires sont disponibles. Müller [Mül92] a montré l'équivalent du résultat de Mc Kean, à savoir qu'il n'y a qu'un nombre fini de surfaces à pointes hyperboliques qui peuvent avoir le même ensemble résonnant.

Par ailleurs, Bérard [Bér92], en raffinant la méthode de Sunada, a pu construire des exemples de surfaces à pointes isospectrales mais non isométriques. Zelditch a aussi construit de tels exemples dans [Zel92].

1.2.3. Ergodicité Quantique

Les questions du Chaos Quantique peuvent être posées dans le cas des variétés à pointes comme dans le cas compact. Il y a plusieurs axes de recherche.

1. Dans le cas des quotients arithmétiques $\Gamma\backslash\mathbb{H}^2$, c'est le spectre discret qui contribue le plus dans la formule de Weyl. De plus, on a alors des opérateurs dit de *Hecke* qui commutent avec le Laplacien. Pour étudier les suites de fonctions propres L^2, on peut se restreindre aux bases qui codiagonalisent le Laplacien et les opérateurs de Hecke. Ce sont pour ces bases-là de fonctions propres que des théorèmes d'unique ergodicité quantique ont été démontrés par Lindenstrauss [Lin06] et Soundararajan [Sou10] (entre autres).

2. Dans le cas où la courbure est variable, il n'y a génériquement qu'un nombre fini de valeurs propres L^2 pour le Laplacien. Il semble donc plus approprié d'étudier les distributions de Wigner pour la famille des $E(s)$. Là encore, plusieurs choix sont possibles.

On peut décider d'étudier les $E(s)$ sur le spectre, i.e pour $\Re s = d/2$. Dans ce cas, pour des surfaces hyperboliques dont le flot géodésique est ergodique, Zelditch [Zel91] a donné un résultat en moyenne.

$$\int_0^T \left| \left\langle \mathrm{Op}_1(\sigma) E\left(\frac{1}{2}+it\right), E\left(\frac{1}{2}+it\right)\right\rangle + \frac{\varphi'}{\varphi}\left(\frac{1}{2}+it\right)\int_{S^*M} \sigma d\mathscr{L}_1 \right| dt = o(T^2). \quad \text{(QEZ)}$$

où $\mathrm{Op}_1(\sigma)$ est un opérateur pseudo-différentiel classique à support compact qui quantifie un symbole σ homogène d'ordre 0. L'intégrale de σ est contre la mesure de Liouville *normalisée* $\mathscr{L}_1$ sur la cosphère unité. Dans le cas (très) particulier de la courbe modulaire, Luo et Sarnak [LS95] ont montré que

$$\left\langle \mathrm{Op}_1(\sigma) E\left(\frac{1}{2}+it\right), E\left(\frac{1}{2}+it\right)\right\rangle = \frac{6}{\pi}|\log t| \int_{S^*M} \sigma d\mathscr{L}_1 + o(\log t). \qquad (1.43)$$

Le fait qu'il n'y ait pas besoin d'une moyenne sur le spectre est véritablement une conséquence du fait que presque tout le «poids» du spectre est discret (autrement dit, $\varphi'/\varphi = o(T)$ sur le spectre).

3. Toujours dans le cas de courbure variable, si on accepte de sortir du spectre, on peut obtenir des résultats sans moyenner. C'est ce qu'a fait Dyatlov [Dya12] pour obtenir

$$\left\langle \mathrm{Op}_1(\sigma) E\left(\frac{1}{2}+\eta+it\right), E\left(\frac{1}{2}+\eta+it\right)\right\rangle = \mu_\eta^\pm(\sigma) + o(1) \qquad (1.44)$$

quand $t \to \pm\infty$ et $\eta > 0$ fixé. Les mesures $\mu_\eta^\pm$ sont supportées sur la sphère. Elles ne sont pas invariantes par le flot, mais vérifient $\mathbf{X}\mu_\eta^\pm = \pm 2\eta\mu_\eta^\pm$, où $\mathbf{X}$ est le champ de vecteur du flot géodésique (autrement dit, elles croissent ou décroissent le long du flot).

1.3. Résultats de la thèse

Durant ma thèse, j'ai travaillé dans plusieurs directions. Dans ce qui suit, (M,g) désigne une variété à pointe.

1.3.1. Ergodicité Quantique

Mon premier travail a été de comprendre une suggestion de Zelditch pour adapter sa preuve d'ergodicité quantique sur le spectre au cas des variétés à pointes dont le flot géodésique est ergodique.

Théorème 1.1. *Soit (M,g) une variété à pointe dont le flot géodésique est ergodique. Alors avec les mêmes notations que pour* (QEZ)*, on a*

$$\int_0^T \left| \left\langle \mathrm{Op}_1(\sigma) E\left(\frac{d}{2}+it\right), E\left(\frac{d}{2}+it\right)\right\rangle + \frac{\varphi'}{\varphi}\left(\frac{d}{2}+it\right)\int_{S^*M}\sigma d\mathscr{L}_1 \right| dt = o(T^{d+1}). \tag{QE'}$$

La preuve de ce résultat est donnée dans le §2.3, dans une formulation semi-classique (théorème 2.5).

Après ce premier résultat, en étudiant la preuve de Dyatlov pour l'estimée (1.44), je me suis rendu compte qu'en utilisant un lemme d'Egorov en temps long, on pouvait étendre (1.44) à des suites de paramètres $s \in \mathbb{C}$ tels que $\Im s \to \pm\infty$, et tels que $\Re s \to d/2$, tant que cette dernière convergence n'était pas trop rapide.

Pour pouvoir appliquer un tel programme, j'ai d'abord construit une quantification Op sur M qui permet de quantifier des symboles qui ne sont pas forcément à support compacts. C'est l'objet des sections 2.1 et 2.2.1.1. Nous n'allons pas décrire dans l'introduction toutes les classes de symboles quantifiées ; La classe qui nous intéresse ici est la suivante.

Si σ est une fonction lisse sur T^*M telle que la norme $|\xi|$ est bornée sur le support de σ, et telle que les dérivées de σ estimées avec la métrique de Sasaki sont toutes bornées, alors $\mathrm{Op}(\sigma)$ est un opérateur pseudo-différentiel semi-classique régularisant.

Pour un tel opérateur, on obtient un lemme d'Egorov en temps long (théorème 2.3) :

$$e^{-ith\Delta}\,\mathrm{Op}(\sigma)e^{ith\Delta} = \mathrm{Op}(\sigma_t) + \mathcal{O}(h^\infty) \tag{1.45}$$

où σ_t est un symbole dans une classe exotique, qui vérifie $\sigma_t = \sigma\circ\varphi_t + \mathcal{O}(h)$ pour des temps bornés. L'asymptotique (1.45) est en fait vrai pour des temps $t < |\log h|/2\lambda_{max}$ où λ_{max} est l'exposant de Lyapunov maximal du flot géodésique (voir définition 2.2.5).

À partir de ce résultat, effectivement, on obtient le fait suivant qui généralise le résultat de S. Dyatlov

Théorème 1.2. *Soit (M,g) une surface à pointe de* courbure strictement négative*. Si $\eta(h)\to 0^+$ avec*

$$\eta(h) \geqslant \lambda_{max}\frac{\log|\log h|}{|\log h|}, \tag{1.46}$$

alors, quand $h \to 0$

$$\eta\left\langle \mathrm{Op}(\sigma)E\left(\frac{d}{2}+\eta(h)+\frac{i}{h}\right), E\left(\frac{d}{2}+\eta(h)+\frac{i}{h}\right)\right\rangle \to \pi\int_{S^*M}\sigma d\mathscr{L}_1. \tag{1.47}$$

Encore une fois, l'intégrale est à comprendre contre la mesure de Liouville normalisée sur la cosphère unité.

Avec les méthodes utilisées dans la preuve de ce théorème, il ne me semble pas possible d'améliorer significativement la borne (1.46). Il serait intéressant de comprendre ce qu'il se passe pour les valeurs de s plus proches de l'axe unitaire $\{\Re s = d/2\}$. L'hypothèse que la courbure est strictement négative est nécessaire pour obtenir la convergence des mesures $\eta\mu_\eta^\pm$ vers la mesure de Liouville, ce qui est une propriété purement classique.

❋

Les résultats rassemblés dans le chapitre 2 (à part l'ergodicité quantique sur le spectre), ont fait l'objet d'une prépublication [Bon14a] sur ArXiV. L'article a été soumis. J'ai reproduit la version soumise, à des modifications mineures près. Il est présenté en anglais, langue dans laquelle il a été écrit au départ.

1.3.2. Une paramétrice pour le déterminant de diffusion

Comme nous l'avons évoqué plus tôt, Selberg a montré en courbure constante, pour les surfaces, que les résonances sont contenues dans une bande. Quand Müller a procédé à une étude systématique de la théorie spectrale du Laplacien pour le cas plus général des surfaces à pointe [Mül92], il a remarqué que ce fait très utile ne semblait pas avoir d'équivalent en courbure variable.

Froese et Zworski [FZ93] ont alors donné un exemple de surface de révolution avec deux pointes pour laquelle la structure de l'ensemble résonnant est très simple. Pour cet exemple, les résonances ne sont pas contenues dans une bande. Néanmoins, c'est un exemple qui présente des ouverts de courbure positive.

Pour tenter d'étendre le résultat de Selberg en courbure variable, il était nécessaire d'en comprendre la preuve. Essentiellement elle repose sur la chose suivante. Le déterminant de diffusion $\varphi(s)$ peut être écrit en courbure constante comme le produit d'un facteur inessentiel et d'une série de Dirichlet. Autrement dit une fonction méromorphe $L(s)$ qui a un développement convergent dans un demi-plan sous la forme

$$L(s) = \sum_{k\geqslant 0} \frac{a_k}{\lambda_k^s}. \tag{1.48}$$

Les λ_k forment une suite strictement croissante de nombres réels. Quand la partie réelle de s est assez grande, c'est le premier terme dans cette expansion qui domine les autres. Ceci implique $L(s)$ ne s'annule pas pour $\Re s$ assez grande, et φ non plus.

Il n'est pas raisonnable d'espérer obtenir une expression exacte comme dans (1.48) dans le cas de la courbure variable. Néanmoins, on peut donner une interprétation dynamique de cette formule. En effet, chaque terme de la somme peut être associé à certaines géodésiques sur la surface qui s'échappent en temps positif et négatif dans les pointes, des *géodésiques diffusées*. Une réinterprétation du résultat de Selberg a été trouvée par Guillemin [Gui77].

Si c est une géodésique diffusée sur M, qui vient de la pointe Z_i en temps négatif, et s'échappe dans Z_j en temps positif, on peut définir le *temps de séjour* de c comme le temps qui s'écoule entre le premier passage en $\{y = a_i\}$ dans Zi, et le dernier passage en $\{y = a_j\}$ dans Z_j. Pour éviter que ceci ne dépende du choix des a_i, on peut choisir une normalisation (voir l'équation (3.10)). On note alors $\mathcal{ST}_{ij}$ l'ensemble des temps de séjour de géodésiques diffusées entre Z_i et Z_j. Dans le chapitre 3, j'obtiens (voir théorème 3.4)

Théorème 1.3. *Soit (M, g) une variété à pointes* de courbure strictement négative. *Il existe un $\delta_g > d/2$ tel que pour $\Re s > \delta_g$, la matrice de diffusion admet le développement convergent suivant pour tout $N \geqslant 1$*

$$\phi_{ij}(s) = \left(\frac{\pi}{s}\right)^{d/2} \left(\sum_{n=0}^{N} \sum_{T \in \mathcal{ST}_{ij}} s^{-n} a_{T,n}^{ij} e^{-sT} + \mathcal{O}(s^{-N} e^{-s\mathcal{T}_{ij}^{\#}}) \right). \tag{1.49}$$

Le temps $\mathcal{T}_{ij}^{\#}$ est une constante géométrique inférieure ou égale au plus petit des temps de séjour.

Dans le cas de la courbure constante, cette écriture est due à Guillemin, mais il obtient une expression sans reste. La constante δ_g est la pression du potentiel $(F^{su} + d)/2$ où F^{su} est le Jacobien Instable. Dans le cas de la courbure constante, $\delta_g = d$. L'hypothèse que la courbure est négative est essentielle pour faire fonctionner la preuve.

Remarque 1.2. *Pour les surfaces à pointes hyperboliques, Zelditch [Zel92] avait montré que les singularités de la transformée de Fourier de la matrice de diffusion sur l'axe $\{\Re s = 1/2\}$ ne pouvait se produire que aux temps de séjours. On peut voir ce résultat comme une conséquence du théorème 1.3, ce qui étend le résultat de Zelditch au variétés à pointes de courbure négatives.*

À partir du théorème 1.3, on peut obtenir un développement en séries de Dirichlet pour φ comme pour ϕ, qui remplace le résultat de Selberg, pour $\Re s > \delta_g$

$$\varphi(s) = \left(\frac{\pi}{s}\right)^{\kappa d/2} \left(L_0(s) + \frac{1}{s} L_1(s) + \cdots + \mathcal{O}(s^{-N} e^{-sT^{\#}}) \right) \tag{1.50}$$

où les L_i sont des séries de Dirichlet

$$L_n(s) = \sum_{k \geqslant 0} \frac{a_k^n}{\lambda_k^s}. \tag{1.51}$$

Il survient alors une difficulté. Les coefficients a_k^n dans le développement en série de φ sont des sommes de nombres qui peuvent être positifs... ou négatifs !

Par conséquent, on peut construire des exemples dégénérés, qui présentent des annulations des coefficients. Nous allons y revenir dans un instant. Mais d'abord

Théorème 1.4. *Soit M une surface à pointe. Pour un ensemble générique de métriques g à courbure strictement négative sur M (ouvert dense en topologie C^2 sur les métriques C^∞), en dehors d'une bande près du spectre, il y a des zones logarithmiques arbitrairement grandes sans résonances. Autrement dit, pour une métrique générique g, il existe $\delta_g' > \delta_g$ telle que pour n'importe quelle constante $C > 0$, l'ensemble des zéros de φ qui sont la région du plan*

$$\left\{\Re s > \delta_g'\right\} \cap \left\{\Re s \leqslant C \log |\Im s|\right\} \tag{1.52}$$

est fini.

De plus, quand il n'y a qu'une pointe, l'hypothèse de généricité n'est pas nécessaire.

Ce résultat repose sur l'argument de Selberg sur les zéros des séries de Dirichlet, et le fait que pour une métrique générique (ou tout le temps quand il n'y a qu'une seule pointe), le coefficient a_0^0 ne s'annule pas.

Proposition 1.3.1. *On peut compléter le théorème 1.4 de trois façons.*

1. *Il y a des exemples de variétés avec* une *pointe, de courbure variable, pour lesquels* toutes *les résonances sont contenues dans une bande verticale.*

2. *Il existe des exemples avec deux pointes qui exhibent des suites de résonances qui se répartissent le long d'une ligne* $\{\Re s \sim -C \log |\Im s|\}$*. En particulier, pour ces exemples, les résonances ne sont* pas toutes *contenues dans une bande.*

3. *Pour un ensemble de métrique ouvert en topologie* C^2*, dont le complémentaire est de codimension infinie en topologie* C^∞*, la paramétrice pour* φ (1.50) *n'est pas nulle, i.e, il y a au moins un des* a_k^n *qui est non-nul. Ceci implique que la conclusion du théorème 1.4 est encore valable, si on remplace* n'importe quelle constante *par* il existe une constante $C > 0$.

Je conjecture

Conjecture 1.1. *L'ensemble de codimension infinie en topologie* C^∞*, est en fait vide. Autrement dit, la paramétrice ne s'annule jamais complètement.*

❄

Les résultats présentés dans le chapitre 3 ont fait l'objet d'un article, prépublié sur internet [Bon15b]. Il n'a pas encore été soumis. Il a été écrit en anglais, langue dans laquelle il a été restranscrit dans la partie 3.

1.3.3. Résultats de Comptage

Au cours de ma thèse, j'ai aussi essayé de voir dans quelles directions on pouvait améliorer les résultats de comptage des résonances. Le premier résultat que j'ai obtenu occupe la section 4.1, qui contient le corps de l'article [Bon14b], prépublié sur ArXiV, et accepté pour publication au *Journal of Spectral Theory.* Il s'agissait d'obtenir l'amélioration (WBR) de (WPR1).

Par ailleurs, dans les travaux de Selberg [Sel89b], on peut trouver des estimées de Weyl très précises (WSR), (WS). On peut aussi y trouver la formule suivante

$$\sum_{\rho\in\mathcal{R},\ -T\leqslant\Im\rho<T} 1 - 2\Re\rho = \frac{\kappa}{\pi}T\log\frac{T}{\pi} - \frac{1}{\pi}(\kappa + 2K)T + \mathcal{O}(\log T). \tag{1.53}$$

Le nombre K est une constante qui dépend de la géométrie de la surface. La preuve de Selberg repose sur des arguments d'analyse complexe et harmonique, qui s'étendent directement au cas de courbure strictement négatives, une fois que l'on dispose de la paramétrice pour φ.

Théorème 1.5. *La formule* (1.53) *est valable dès que la paramétrice est non-nulle, c'est à dire pour un ensemble ouvert et dense en topologie* C^∞ *sur les métriques de courbure strictement négative. Dans ce cas, on a aussi*

$$\sum_{\substack{\rho\in\mathcal{R},\ |\rho|\leqslant T\\ \Re\rho\leqslant d-\delta_g'<d/2}} 1 = \mathcal{O}(T). \tag{1.54}$$

Les exemples donnés dans le point **2.** *de la proposition 1.3.1 saturent cette borne.*

La preuve de ce résultat apparait dans la partie 4.2, qui est une note, prépubliée sur ArXiV, elle aussi en Anglais.

L'estimée en $\mathcal{O}(T)$ correspond à une estimée de comptage pour les résonances d'un problème de diffusion unidimensionnel. Cette estimée était en fait satisfaite dans l'exemple de Froese et Zworski.

Je pense que les arguments de Selberg qui lui permettent de démontrer (WSR) devraient pouvoir être adaptés au cas courbure négative stricte, si l'on dispose par ailleurs d'une estimée aussi bonne que (WS). L'estimée en question semble pouvoir être obtenue en utilisant les méthodes de Bérard [Bér77]. Je n'ai malheureusement pas eu le temps d'investiguer plus avant ce sujet.

❇

Concluons cette introduction avec l'observation que les articles, à part des modifications mineures de notations, sont reproduits in-extenso, et sont donc autocontenus.

Chapitre 2

Un résultat d'unique ergodicité quantique

Dans ce chapitre, je reproduis le contenu de l'article [Bon14a], qui a été soumis. Dans une première partie, je donne une construction pour une quantification et une classe de symboles adaptés aux pointes hyperboliques. Dans la deuxième partie, je m'en sers pour donner un lemme d'Egorov jusqu'au temps d'Ehrenfest. Ce résultat est utilisé pour généraliser le résultat de S. Dyatlov 1.44 à des suites de paramètres s qui se rapprochent de l'axe unitaire, mais pas trop vite. J'obtiens le théorème 2.4.
Au début de ma thèse, j'ai utilisé une idée de S. Zelditch pour généraliser son théorème d'Ergodicité Quantique QEZ au cas des variétés à pointes dont le flot est ergodique (sans hypothèse de courbure négative ni constante). En le contactant de nouveau il m'a communiqué une autre preuve, utilisant une autre de ses idées, qui était bien plus courte que la mienne. J'en donne le détail dans la partie 2.3
Les appendices originellement liés à l'article ont été disposés avec les autres appendices à la fin de ce document.

❄

In this paper, we work with non-compact complete manifolds (M, g) of finite volume with hyperbolic ends. Such manifolds are called *cusp manifolds.* They decompose into a compact manifold with boundary and a finite number of cusp-ends $Z_1, \dots Z_\kappa$, that is, of the type:

$$Z_a^\Lambda = [a, +\infty)_y \times \mathbb{T}^d_\theta \text{ with the metric } ds^2 = \frac{dy^2 + d\theta^2}{y^2}$$

where $d\theta^2$ is the canonical flat metric on the d-dimensionnal torus $\mathbb{T}^d = \mathbb{R}^d/\Lambda$. The Laplacian on compactly supported smooth functions on M is essentially self-adjoint, so it has a unique self-adjoint extension Δ to $L^2(M)$. Here, we take the analyst's convention that $-\Delta \geqslant 0$. In [CdV83], Yves Colin de Verdière proved that for cusp *surfaces*, the resolvent of the Laplacian has a meromorphic continuation through the continuous spectrum. Another proof was given in any dimension with a more general definition of cusps by Müller in [Mül83]. It gives the following. The operator defined on $L^2(M)$, for $\Re s > d/2$ and $s \notin [d/2, d]$, by

$$\mathcal{R}(s) = (-\Delta - s(d-s))^{-1}$$

has a meromorphic continuation to the whole complex plane, as an operator $C_c^\infty(M) \to C^\infty(M)$. The set of poles is called the *resonant set*. The poles on the vertical line $\Re s = d/2$ (called the *unitary axis*), and on the segment $[0, d]$ correspond to discrete L^2 eigenvalues. However, the others are associated to the continuous spectrum and called *resonances*. The way to prove this uses a meromorphic family of eigenfunctions, the so-called *Eisenstein series* $\{E_i(s)\}_{s\in\mathbb{C}, i=1\dots\kappa}$. Those are smooth, *not* L^2, and satisfy

$$-\Delta E_i(s) = s(d-s)E_i(s), \quad s \in \mathbb{C}.$$

The $E_i(s)$ naturally replace the L^2 eigenfunctions as spectral data for the continuous spectrum. Actually, the data can be alternatively considered to be the values of $E(s) = (E_1, \dots, E_\kappa(s))$ for s on the unitary axis, the full family $\{E(s)\}_{s\in\mathbb{C}}$, or the poles and the associated residues of the family — a.k.a the *resonant states*, see definition 2.2.6.

Observe that it is not always easy to translate results between those formulations. We are trying to determine whether properties of L^2 eigenfunctions for compact manifolds remain true for these spectral data in the non-compact case; we also seek to know if new behavior arise. In particular, we study some measures associated to sequences of spectral parameters, called semi-classical measures.

In this paper, we build a semi-classical pseudo-differential calculus $\Psi(M)$ with symbols $S(M)$ and a quantization Op_h (sections 1 and 2.1). For $s \in \mathbb{C}$ not a resonance, let $\mu_i^h(s)$ be the distribution on T^*M

$$\langle \mu_{i,j}^h(s), \sigma \rangle := \langle \mathrm{Op}_h(\sigma) E_i(s), E_j(s) \rangle, \ \sigma \in C_c^\infty(T^*M).$$

Also consider

$$\langle \mu^h(s), \sigma \rangle := \sum_i \langle \mathrm{Op}_h(\sigma) E_i(s), E_i(s) \rangle.$$

In the case of surfaces, Semyon Dyatlov [Dya12] proved that when $|\Im s| \to \infty$, with $h = 1/|\Im s|$ and $\Re s \to 1/2 + \eta$ with $\eta > 0$, one has

$$\eta \mu_{i,i}^h \rightharpoonup \mu_{i,\eta}. \tag{2.1}$$

The measure $\mu_{i,\eta}$ is an explicit measure on S^*M — the unit cotangent sphere. It satisfies $(X - 2\eta)\mu_{i,\eta} = 0$ where X is the generator of the geodesic flow. This result does not rely upon dynamical properties of the geodesic flow such as ergodicity. We recover a similar result in any dimension — theorem 2.4. When the curvature of the manifold is (strictly) negative, we adopt Babillot's argument [Bab02] which relies on the Local Product Structure to prove that

$$\mu_{i,\eta} \underset{\eta\to 0}{\rightharpoonup} \mathscr{L}_1 \qquad \text{(lemma 2.2.13)}$$

where $\mathscr{L}_t$ is the normalized Liouville measure on tS^*M . One can wonder whether Dyatlov's result still holds when $\Re s \to 1/2$, replacing $\mu_{i,\eta}$ by the Liouville measure. We prove the following weak Quantum Unique Ergodicity (QUE) result:

Theorem 2.1. *Assume M is a cusp manifold of negative curvature. Then there is a positive constant C_0 (only depending on the maximal Lyapunov exponent of the geodesic flow on the manifold, see definition 2.2.5) such that the following holds. Assume that*

$$\eta(h) \underset{h\to 0}{\to} 0 \text{ and } \eta > C_0 \frac{\log|\log h|}{|\log h|}. \tag{2.2}$$

Then,

$$\eta \times \mu^h_{i,j}\left(\frac{d}{2} + \eta + \frac{i}{h}\right) \underset{h\to 0, \eta\to 0}{\longrightarrow} \delta_{ij}\pi \mathscr{L}_1$$

The condition in equation (2.2) seems to be essential to the type of arguments used in the proof.

It is worthwile to recall the result obtained by Steven Zelditch, in [Zel91], of Quantum Ergodicity for $E(s)$ with s on the unitary axis, in the case of *hyperbolic* surfaces. The set of poles of $\{E(s)\}_{s\in\mathbb{C}}$ is encoded in what is called the *Scattering phase*, which is a function $\mathcal{S}$ on $\mathbb{R}$ (see section 2.1.1).

Theorem (Zelditch)**.** *For any $T > 0$,*

$$h \int_{-T}^{T} \left| \mu^h\left(\frac{d}{2} + \frac{it}{h}\right) - 2\pi \mathcal{S}'\left(\frac{t}{h}\right)\mathscr{L}_{t^2}\right| dt \underset{h\to 0}{\longrightarrow} 0$$

We called our result a *QUE result* because we obtain a convergence, and not a *Cesaro* convergence as in Zelditch's theorem. However, it is a *weak* statement because we obtain information for spectral parameters s that are not on the unitary axis $\{\Re s = d/2\}$, i.e on the spectrum, or arbitrarily close to it. A *strong* QUE result would be to remove the $\log\log/\log$ condition on $\eta(h)$ — or at least improve it. It is not quite clear what the resulting asymptotics would be, and it is probably a difficult problem.

This can be contrasted with the convex cocompact case — replacing cusps by funnels — for which results have recently appeared — see [GN14] and [DG14].

The only cusp surface, as far as we know, for which such results have been obtained is the *modular curve* — it as actually an orbifold. Luo and Sarnak proved in [LS95] that

$$\mu^h\left(\frac{1}{2} + \frac{i}{h}\right) \sim \frac{6}{\pi}|\log h| \mathscr{L}_1. \tag{2.3}$$

Recently [PRR11], Petridis, Raulf and Risager extended this to

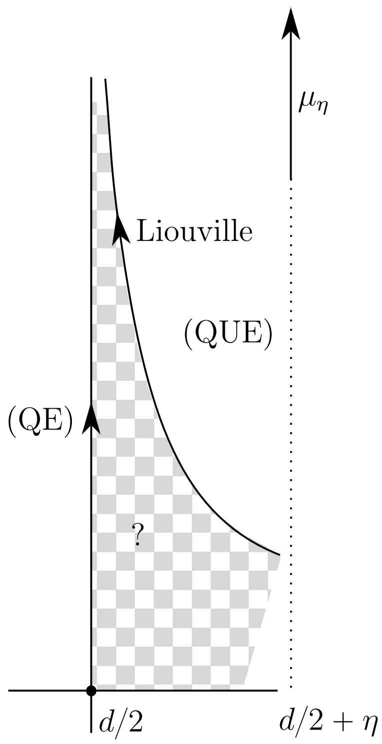
$$\mu^h\left(\frac{1}{2} + \eta + \frac{i}{h}\right) \sim \frac{6}{\pi}|\log h| \mathscr{L}_1. \tag{2.4}$$

as long as $\eta|\log h| \to 0$. However, the case of the modular surface, for which most of the spectrum is discrete, is very particular. There is no reason to expect that (2.3) or (2.4) should hold in general for other surfaces.

Let us also remark that we obtain information on the microlocal distribution of resonant states. If s is a pole of E, then resonant states at s are some linear combinations of the E_i's at $d-s$ — see definition 2.2.6. Assume that a sequence of resonances $s(h)$ is such that $d-s$ satisfies the hypothesis of theorem 2.1, and take a corresponding sequence u_h of resonant states. In the same way we defined the Wigner distributions $\mu^h_{i,j}$ for $E_i(s)$, we can define a sequence of Wigner distributions ν_h associated to u_h. Since the normalization of the resonant state is not canonical, it is harmless to renormalize the ν_h. The theorem shows that we can do it so that the limit is the Liouville measure.

Remark Apart from the case of arithmetic cusp surfaces, it is quite possible that the region of the plane of theorem 2.1 (above the curve in the figure) contains *no* resonances. More generally, $\eta \to \nu > 0$ corresponds to a negligible part of the resonances, and we suspect it is also the case of resonances satisfying equation (2.2), at least for generic metrics.

The figure on the right gives a synthetic vision of what is known (including our result) of the semi-classical measures: arrows represent asymptotics of sequences of s's, and we give the associated semi-classical measure next to it, as we explain below.



The main tool that enables us to extend Dyatlov's theorem is a long time Egorov lemma, for the purpose of which we constructed our calculus $\Psi(M)$. Let p be the principal symbol of $-h^2\Delta/2$, that is, the metric $p(x,\xi) := |\xi|^2_x/2$.

Theorem 2.2. *Let $\sigma \in S(M)$ be supported in a set where p is bounded. There is a symbol σ_t in an exotic class such that for $|t| \leqslant C|\log h|$*

$$e^{-ith\Delta}\operatorname{Op}(\sigma)e^{ith\Delta} = \operatorname{Op}(\sigma_t) + \mathcal{O}(h^\infty).$$

This holds as long as $C > 0$ remains smaller than $C_{max} = 1/2\lambda_{max}$ where λ_{max} is the maximal Lyapunov exponent of the geodesic flow on M.

As in most situations, the proof of our Egorov lemma is relatively easy once the settings, and in particular, the relevant properties of the quantization have been established. Let us explain; apart from some abstract nonsense, the analysis in the Egorov lemma is contained in an estimate of the derivatives of the flow. In the compact case, the choice of how those derivatives are estimated is not important. However, in a non-compact case, one has to use norms consistent with the geometry of the problem, and those norms determine the class of symbols to use.

One way to avoid those problems is to use a compactly supported quantization and operators. This is adapted in cases where the whole interesting part of the underlying dynamics takes place in a compact set (see for example [DG14]). However in our case, a crucial part of the dynamics happens in the cusps, so we wanted to allow for symbols supported in the cusps, and we had to give a treatment for the ends.

Another description we found was in [Bou14]. For symbols only depending on the vertical variables in the cusps, this is probably easier to manipulate. However it does not allow for flexible behavior on the θ variable. Let us observe however that [Bou14] deals with cusps that are much more general than the ones we consider.

We found a only closely related description of pseudo-differential operators in [Zel86] by Zelditch. He develops a quantization procedure for the hyperbolic plane, using Fourier-Helgason transform on the unit disk. Then, he shows that those operators can be symmetrized to operate on compact hyperbolic surfaces. In some sense, we use *the* class of symbols that are compatible with the metric, and that class is similar to that introduced by Zelditch.

However, the quantization procedure is different: we use only euclidean Fourier transform to build operators specifically on cusps — see equation (2.7). We prove composition stability without any proper support assumption. This is available in any dimension $\geqslant 2$, in a semi-classical formulation. We also state usual theorems on pseudo-differential operators (pseudors), including L^2 bounds, and a trace formula. We detailed the proofs, with elementary tools, mostly referring to [Zwo12].

It would be interesting to see how this calculus fits in the *cusp*-calculus theory developped in [MM98]

This will be part of the author's PhD thesis.

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Organization of the paper Section 1 is devoted to describing a quantization procedure on cusps, proving it enjoys all usual elementary properties. In section 2.1, the results of section 1 are used to construct a quantization on cusp manifolds that enables one to quantize symbols that are supported in the whole manifold; the Egorov lemma is given is section 2.1.2. At last, we deduce the main theorem on semi-classical measures in section 2.2.

2.1. Quantizing in a full cusp

2.1.1. Symbols

Let Z_Λ be a *full cusp*. That is

$$Z_\Lambda = \mathbb{R}^+ \times \mathbb{R}^d/\Lambda,$$

where Λ is some lattice in $\mathbb{R}^d$. The first coordinate, y, is called the *height* ; the second is denoted by θ, and we write $x = (y, \theta) = (y, \theta_1, \dots, \theta_d)$. The cusp is endowed with a *cusp metric* :

$$ds^2 = \frac{dy^2 + d\theta^2}{y^2},$$

where $d\theta^2 = d\theta_1^2 + \cdots + d\theta_d^2$ is the canonical flat metric on $\mathbb{R}^d$. By rescaling, we see that Z_Λ and $Z_{t\Lambda}$ are isometric whenever $t > 0$, so we assume that Λ has covolume 1. In the first part of the article, we will write just Z for Z_Λ because Λ will not change. The Riemannian volume in Z is

$$d\mathrm{vol}(x) = \frac{dyd\theta}{y^{d+1}}.$$

We refer to the space of square integrable functions with respect to this volume as $L^2(Z)$. The Laplacian is

$$\Delta = y^2\Delta_{\mathrm{eucl}} - (d-1)y\frac{\partial}{\partial y},$$

where Δ_{eucl} is the Laplacian for the Euclidean cylinder. On the cotangent bundle T^*Z, we let Y and J be the dual coordinates to ∂_y and ∂_θ, with $\xi = Ydy + Jd\theta$ (J is a vector in $\mathbb{R}^d$). We also write $\xi = (Y, J) = (Y, J_1, \ldots, J_d)$. Letting ∇ be the usual flat connexion on $\mathbb{R}^d$, the Poisson Bracket on T^*Z writes

$$\{f, g\} = \partial_Y f \partial_y g - \partial_y f \partial_Y g + \nabla_J f.\nabla_\theta g - \nabla_\theta f.\nabla_J g.$$

The riemannian metric on Z gives an isomorphism between TZ and T^*Z, and T^*Z is thus a Euclidean fiber bundle endowed with the metric

$$|\xi|_x^2 = y^2(Y^2 + |J|^2).$$

In appendix A.2, we recall how to define the spaces $\mathscr{C}^n(Z)$ of functions using covariant derivatives. This definition is intrinsic of the metric, however it is not very practical for computations. Let

$$X_y := y\partial_y \quad X_{\theta_i} := y\partial_{\theta_i}.$$

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a sequence with $\alpha_i \in \{y, \theta_1, \ldots, \theta_d\}$ — we say a *space-index of length* k — we denote $X_{\alpha_1} \ldots X_{\alpha_k}$ by X_α, and $|\alpha| = k$ is the length. Then we prove in (A.2) that

$$\|.\|_{\mathscr{C}^n(Z)} \text{ and } \sum_{|\alpha| \leqslant n} \|X_\alpha f\|_{L^\infty(Z)} \text{ are equivalent norms.}$$

Since $[X_y, X_\theta] = X_\theta$, the order in which the derivatives are taken has little importance.

Definition 2.1.1. *We call* coefficients *the elements of* $\mathscr{C}_b^\infty(Z) = \cap_{n \geqslant 0} \mathscr{C}^n(Z)$, *that is, smooth functions* f *on* Z *such that for any space-index* α,

$$\|X_\alpha f\|_{L^\infty(Z)} < \infty.$$

Elements of $\mathscr{C}_{b,h}^\infty(Z) := C^\infty(\mathbb{R}^+ \to \mathscr{C}_b^\infty(Z))$ *are called* (semi-classical) *coefficients.*

Both $\mathscr{C}_b^\infty(Z)$ and $\mathscr{C}_{b,h}^\infty(Z)$ have a natural ring structure.

Lemma 2.1.2. *The module generated by* X_y *and* X_θ*'s over the coefficients* $\mathscr{C}_b^\infty(Z)$ *is a Lie algebra.*

Proof. It suffices to consider the behavior of $[aX_y, bX_{\theta_i}]$ with a, b in $\mathscr{C}_b^\infty(Z)$:

$$[aX_y, bX_{\theta_i}] = (ab + aX_y b)X_{\theta_i} - (bX_{\theta_i} a)X_y$$

□

If we consider hX_y and hX_{θ_i}'s as vector fields on Z with a parameter $h \geqslant 0$, we deduce that the module they generate over $\mathscr{C}_{b,h}^\infty(Z)$ is also a Lie algebra. Hence it makes sense to speak of its universal enveloping algebra $\mathcal{V}(Z)$. This is an algebra of *semi-classical differential operators* on Z. From now on, all differential operators we use will be in $\mathcal{V}(Z)$.

Inside of $\mathcal{V}(Z)$, we can consider the subalgebra generated by hX_y and hX_{θ_i}'s over $\mathbb{C}$; those are the *constant coefficients* differential operators. It is easy to check

Proposition 2.1.3. *The elements of $\mathcal{V}(Z)$ can be decomposed as finite sums of the type*

$$\sum_\alpha h^{|\alpha|} a_\alpha(h,x) X_\alpha$$

with $a_{i,\alpha}$'s in $\mathscr{C}^\infty_{b,h}(Z)$.

We define :

$$P = -\frac{h^2}{2}\Delta.$$

It is in $\mathcal{V}(Z)$ and in some sense, $\mathcal{V}(Z)$ has been taylored to satisfy this property. Actually, P is a constant coefficient operator:

$$P = -\frac{1}{2}\left((hX_y)^2 + (hX_{\theta_1})^2 + \dots + (hX_{\theta_d})^2\right) + h\frac{d}{2} \times hX_y.$$

Using the algebraic properties described above, one can prove:

Proposition 2.1.4. *Let $A \in \mathcal{V}(Z)$, and $(x,\xi) \in T^*Z$. Let ϕ be a smooth function on Z, with $\phi(x) = 0$ and $d\phi(x) = \xi$. Then we let*

$$\sigma^0(A)(x,\xi) = \lim_{h\to 0} A_h(e^{i\phi/h})(x).$$

This limit exists and does not depend on the choice of ϕ. It is called the principal symbol *of A. It is a polynomial in the ξ variable, smoothly depending on x and h. What is more, its monomials are of the form $a(y,\theta)y^{k+l}Y^k J^l$ where $a \in C^\infty_b(Z)$.*

The mapping σ^0 from $\mathcal{V}(Z)$ to functions on T^*Z is linear. What is more, if A, B are in $\mathcal{V}(Z)$,

$$\sigma^0(AB) = \sigma^0(A)\sigma^0(B) \quad \textit{and} \quad \sigma^0\left(\frac{1}{h}[A,B]\right) = \left\{\sigma^0(A), \sigma^0(B)\right\}$$
$$\sigma^0\left(\frac{h}{i}X_y\right) = yY \quad \textit{and} \quad \sigma^0\left(\frac{h}{i}X_{\theta_i}\right) = yJ_i$$

We deduce that the principal symbol of P is the metric

$$p = |\xi|^2_x/2. \tag{2.5}$$

If we consider the set of functions $S_{\mathcal{V}}$ on T^*Z obtained by taking principal symbols of differential operators, we see that it is stable by the action of the vectors X_y and X_{θ_i}'s. In the ξ direction, we introduce the dual fields

$$X_Y := \frac{1}{y}\partial_Y \quad X_{J_i} := \frac{1}{y}\partial_{J_i}.$$

Then X_Y and X_{J_i} also stabilize $S_{\mathcal{V}}$. Let us refine this description. $S_{\mathcal{V}}$ is a graded algebra decomposing as $S_{\mathcal{V}} = \cup_{n\geqslant 0} S^n_{\mathcal{V}}$, where $S^n_{\mathcal{V}}$ is the set of q's of degree lesser than n in ξ. Then X_y and X_θ's map $S^n_{\mathcal{V}}$ to itself while X_Y and X_J's map $S^n_{\mathcal{V}}$ to $S^{n-1}_{\mathcal{V}}$. Also remark that whenever $q \in S^n_{\mathcal{V}}$, $q = \mathcal{O}(\langle y\xi\rangle^n)$ where $\langle x\rangle$ is the usual bracket $\sqrt{1+x^2}$.

The above motivates the introduction of the following class of symbols:

Definition 2.1.5. *Let σ be a smooth function on T^*Z. We say that σ is a* (hyperbolic) symbol *on Z of order $n \in \mathbb{R}$ if for any finite sequence $\{\alpha_k\}$ with $\alpha_k \in \{y, \theta_{1\ldots d}, Y, J_{1\ldots d}\}$, if (α) is the number of Y's and J's in the sequence,*

$$q_{n,\alpha} := \sup_{T^*Z} \langle y\xi\rangle^{(\alpha)-n} |X_{\alpha_1} \ldots X_{\alpha_k} a(x,\xi)| \leqslant C. \tag{2.6}$$

We let $S(Z)$ be the set of symbols, $S^n(Z)$ the set of symbols of order n, and $S^{-\infty} = \cap_{n\in\mathbb{R}} S^n(Z)$. S is graded by the order, and

$$\begin{aligned} S^n_{\mathcal{V}} &\subset S^n(Z) \\ X_y, X_{\theta_i} &: S^n(Z) \to S^n(Z) \\ X_Y, X_{J_i} &: S^n(Z) \to S^{n-1}(Z) \\ \sigma \in S^m, \mu \in S^n &\Rightarrow \{\sigma, \mu\} \in S^{m+n-1} \end{aligned}$$

The family of semi-norms $q_{n,\alpha}$ gives a structure of metric space to S^n.

We have not specified an h-dependency. Actually, we will need to let symbols depend on h in a slightly rough fashion, so we use the *exotic* classes :

Definition 2.1.6. *Let $0 \leqslant \rho < 1/2$. Consider complex functions σ of $h > 0$ and $(x,\xi) \in T^*Z$ such that for fixed h, $\sigma_h = \sigma(h,\cdot)$ is in S^n. If α is a sequence of indices, let $|\alpha|$ be its length. Assume that σ additionnally satisfy the family of estimates*

$$q_{n,\alpha,\rho} := \sup_h h^{\rho|\alpha|} q_{n,\alpha}(\sigma_h) < \infty, \text{ for } \alpha \text{ finite sequence of } y, \theta_i, Y, J_i.$$

Then we say that σ is an exotic symbol of order (n,ρ), and write $\sigma \in S^n_\rho(Z)$. We also define $S_\rho(Z) = \cup S^n_\rho(Z)$ and $S^{-\infty}_\rho(Z) = \cap S^n_\rho(Z)$. We have:

$$\begin{aligned} h^\rho X_y, h^\rho X_{\theta_i} &: S^n_\rho(Z) \to S^n_\rho(Z) \\ h^\rho X_Y, h^\rho X_{J_i} &: S^n_\rho(Z) \to S^{n-1}_\rho(Z) \\ \sigma \in S^m_\rho, \mu \in S^n_\rho &\Rightarrow h^{2\rho}\{\sigma, \mu\} \in S^{m+n-1}_\rho \end{aligned}$$

The semi-norms $q_{n,\alpha,\rho}$ also give a structure of metric space to S^n_ρ.

The rest of section 1 is devoted to describing a quantization procedure for this algebra of symbols. In section 1.2, we give a definition, and prove that we obtain pseudo-differential operators (pseudors) in the usual sense. Then, in section 1.3, we give a stationary phase lemma. We use it to prove usual properties of the quantization in section 1.4.

2.1.2. Quantizing symbols.

From now on, we will write S, S^n, ... instead of $S(Z)$, $S^n(Z)$, ...

After we give a quantization procedure Op in 2.1.7 for $\sigma \in S_\rho$, we first prove (lemma 2.1.8) that the operators we obtain have pseudo-differential behaviour locally. That is, if γ is a diffeomorphism from a relatively compact open set U in $\mathbb{R}^n$ onto its relatively compact image in Z, the pullback

$$(\gamma^* \operatorname{Op}(\sigma))f(x) = [\operatorname{Op}(\sigma)(f \circ \gamma)](\gamma(x)).$$

is a pseudo-differential operator in U. Then we prove (lemma 2.1.15) that Op is *pseudo-local* in the following sense : if ϕ_1 and ϕ_2 are two coefficients on Z not depending on θ, with disjoint support,

$$\phi_1 \operatorname{Op}(\sigma)\phi_2 = O_{H^{-n}\to H^n}(h^\infty) \quad \text{for every } n \in \mathbb{N}.$$

The Sobolev spaces on cusps are defined in appendix A.2. To get to lemma 2.1.15, we have to prove that our operators have *some* Sobolev regularity — see proposition 2.1.12. This is deduced, with a usual parametrix argument, from the crucial lemma 2.1.9 about regularity on $L^2(Z)$. The regularity we prove in this section is certainly not optimal, and we will get better statements — see proposition 2.1.22 — once we obtain stability by composition.

For convenience, we will use some expressions in the half-space $\mathbb{H}^{d+1} = \mathbb{R}^+ \times \mathbb{R}^d$, which is the universal cover of Z. If f is some function on Z (resp. T^*Z), we identify it with its unique lift to $\mathbb{H}^{d+1}$ (resp. $T^*\mathbb{H}^{d+1}$). We denote by $\mathbf{Op}_h^w$ the *usual Weyl quantization* of a symbol on $\mathbb{R}^{d+1}$:

$$\mathbf{Op}_h^w(\eta)f = \frac{1}{(2\pi h)^{d+1}} \int_{\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}} e^{i\langle x-x',\xi\rangle/h} \eta\left(\frac{x+x'}{2}, \xi\right) f(x')dx'd\xi. \tag{2.7}$$

We are only interested by the restriction of this expression to the upper half space. If η and f are only defined on the upper half space, this also makes sense (continuing all functions by 0 in the lower half space). Now, assume that η is in S^n_ρ, and f is a smooth compactly supported function on Z — that is a periodic function in θ in $\mathbb{H}^{d+1}$, with compact support in $0 < y_- \leqslant y \leqslant y_+ < +\infty$. Then, in each fiber of T^*Z, η is a symbol in the usual sense of $\mathbb{R}^n$, with estimates uniform as long as y stays in a compact set of $\mathbb{R}^{+*}$. We deduce that the Fourier transform of η fiberwise is a distribution in $\mathscr{S}'$, whose singular support is $\{0\}$ and with fast decay at infinity, with estimates uniform in $x = (y, \theta)$ as long as y stays in a compact set. We deduce that $\mathbf{Op}_h^w(\eta)f$ is well defined in the upper half space. Actually, applying a finite number of X_y's and X_θ's, we can repeat the argument and observe it is also a smooth function. A similar argument will be detailed in the proof of lemma 2.1.9. Now, we can give our basic definition :

Definition 2.1.7. *Let $\sigma \in S_\rho(Z)$ be some hyperbolic symbol on T^*Z. Then, for any $f \in C_c^\infty(Z)$, we let for $y > 0$*

$$\operatorname{Op}_h(\sigma)f(y,\theta) = y^{\frac{d+1}{2}} \mathbf{Op}_h^w(\sigma)\left(\frac{1}{y^{\frac{d+1}{2}}} f\right)(y,\theta).$$

This is seen to be a Λ-periodic function in the θ direction. $\operatorname{Op}_h(\sigma)$ *defines an operator $C_c^\infty(Z) \to C^\infty(Z)$. When σ is a symbol,* $\operatorname{Op}(\sigma)$ *denotes the family* $\{\operatorname{Op}_h(\sigma)\}_h$.

We let Ψ^n_ρ be the set of $\{\operatorname{Op}(\sigma) | \sigma \in S^n_\rho\}$.

The introduction of $(y/y')^{(d+1)/2}$ corresponds to the conjugacy by the unitary map

$$\mathcal{L} : f \mapsto y^{(d+1)/2} f \text{ from } L^2(dyd\theta) \text{ to } L^2(Z). \tag{2.8}$$

2.1.2.1. The principal symbol

The exotic symbols do not necessarily have a limit when h goes to 0, so we have to change somewhat the definition of the principal symbol. Actually, if ϕ is as in proposition 2.1.4, and $A \in \mathcal{V}^n(Z)$,

$$A(e^{i\phi/h})(x) = \sigma^0(x,\xi) + \mathcal{O}(hS^{n-1}_{\mathcal{V}})(x,\xi). \tag{2.9}$$

The remainder actually depends on the choice of ϕ. So we can see $\sigma^0(x,\xi)$ as an element of $S^n_{\mathcal{V}}/hS^{n-1}_{\mathcal{V}}$.

Lemma 2.1.8. *The definition of the semi-classical principal symbol given for elements of $\mathcal{V}(Z)$ as elements of S^n/hS^{n-1} extends to operators* $\mathrm{Op}(\sigma)$*'s with $\sigma \in S(Z)$, and $\sigma^0(\mathrm{Op}(\sigma)) = \sigma$. When $A \in \Psi^n_\rho$, $\sigma^0(A)$ is defined as an element of $S^n_\rho/h^{1-2\rho}S^{n-1}_\rho$.*

Proof. Let χ be some $C^\infty_c(Z)$ function equal to 1 near $x \in Z$. Let $\xi_0 \in T^*_x Z$ and ϕ some smooth function on Z such that $\phi(x) = 0$ and $d\phi(x) = \xi_0$. Let $\sigma \in S_\rho$, take $\tilde{x}$ a lift of x in $\mathbb{H}^{d+1}$ and integrating over $\mathbb{H}^{d+1} \times \mathbb{R}^{d+1}$

$$\mathrm{Op}_h(\sigma)\left(\chi e^{i\phi/h}\right)(x) = \frac{1}{(2\pi h)^{d+1}} \int e^{i(\langle \tilde{x}-x',\xi\rangle + \phi(x'))/h} \left(\frac{y}{y'}\right)^{(d+1)/2} \sigma\left(\frac{\tilde{x}+x'}{2},\xi\right)\chi(x')dx'd\xi$$

The proof here will be very similar to the case of $\mathbb{R}^n$, so we just insist on the differences. The point is to show that up to a remainder $\mathcal{O}(h^{1-2\rho})$, the first order asymptotics does not depend on the choice of ϕ. We let $\tilde{\chi}$ be some smooth compactly supported function equal to 1 around $\tilde{x}$; we insert $1 = \tilde{\chi} + (1-\tilde{\chi})$ to break the integral into two parts (I) and (II).

For the first term (I), observe that it is an integral over a fixed compact set in the x' variable, with an integrand that has symbolic behavior in the ξ variable, and a very simple phase function. Classical stationary phase results directly apply to give that :

$$(\mathrm{I}) = \sigma_h(x,\xi_0) + \mathcal{O}(h^{1-2\rho}S^{n-1}_\rho)(x,\xi_0).$$

To estimate (II), we integrate by parts in ξ; we just have to introduce suitable powers of $(y+y')$ to obtain the new integrand

$$C_k h^{2k-d-1} e^{i(\langle \tilde{x}-x',\xi\rangle+\phi(x'))/h} \left(\frac{y}{y'}\right)^{(d+1)/2} \frac{(y+y')^{2k}}{|\tilde{x}-x'|^{2k}}(1-\tilde{\chi}(x'))\sigma^*_k\left(\frac{\tilde{x}+x'}{2},\xi\right)\chi(x'),$$

where $\sigma^*_k = (X_Y^2 + X_J^2)^k\sigma$ and $C_k = (i/2)^{2k-d-1}\pi^{-d-1}$. With k big enough, this is $\mathcal{O}(h^{(1-\rho)2k-d-1})$ in $L^1(dx'd\xi)$.

□

2.1.2.2. Basic boundedness estimates.

Lemma 2.1.9. *For all $\epsilon > 0$, the elements of $\Psi^{-d-1-\epsilon}_\rho$ extend to bounded operators on $L^2(Z)$, with $\mathcal{O}(h^{-\rho d-1})$ norm as $h \to 0$.*

In the subsequent developments, we will call *Schwarz Kernel* of $A : C^\infty_c \to C^\infty$ the distribution K defined by :

$$Af(x) = \int_Z K(x,x')f(x')dx'$$

where $dx' = dyd\theta$ is the Lebesgue measure in the half-cylinder. Let us state a modified Schur inequality — see the proof of theorem 4.21, page 82, in [Zwo12] for the original version.

Lemma 2.1.10. *Let A be an operator from $C_c^\infty(Z)$ to $C^\infty(Z)$, with Schwarz kernel K. Assume that for some $\tau \in \mathbb{R}$,*

$$C(A,\tau) := \sup_x \int_{x'\in Z} \left(\frac{y'}{y}\right)^{d+1+\tau} |K(x,x')|dx' \times \sup_{x'} \int_{x\in Z} \left(\frac{y}{y'}\right)^{\tau} |K(x,x')|dx < \infty. \quad (2.10)$$

Then A can be extended to a bounded operator on $L^2(Z)$ with

$$\|A\|^2_{L^2\to L^2} \leqslant C(A,\tau).$$

Proof. We follow the classical proof. All the integrals are over Z. if $u \in C_c^\infty(Z)$,

$$\begin{aligned}
|Au(x)| &\leqslant \int \sqrt{y'^{d+1+\tau}|K|}\sqrt{y'^{-d-1-\tau}|K||u|^2}dx' \\
&\leqslant \|y'^{d+1+\tau}|K|\|^{1/2}_{L^1(x')} \left[\int y'^{-\tau}|K(x,x')||u|^2 y'^{-d-1}dx'\right]^{1/2}.
\end{aligned}$$

$$\begin{aligned}
\int |Au(x)|^2 y^{-d-1}dx &\leqslant \left[\sup_x \int_{x'} \left(\frac{y'}{y}\right)^{d+1+\tau} |K(x,x')|\right] \int_{x,x'} \left(\frac{y}{y'}\right)^{\tau} |K(x,x')| \frac{|u|^2}{y'^{d+1}} \\
&\leqslant C(A,\tau) \int \frac{|u|^2 dx'}{y'^{d+1}}.
\end{aligned}$$

□

Now, we prove lemma 2.1.9

Proof. Let $\sigma \in S_\rho^{-d-1-\epsilon}$ with some $\epsilon > 0$. Formula (2.7) actually defines $\mathbf{Op}^w(\sigma)$ acting on the half plane. We let $K_\sigma^w(x,x')$ be its kernel, and we let K_σ be the kernel of $\mathrm{Op}_h(\sigma)$ — K_σ depends on h. Then we have

$$K_\sigma(y,\theta,y',\theta') = \left(\frac{y}{y'}\right)^{\frac{d+1}{2}} \sum_{k\in\Lambda} K_\sigma^w(y,\theta,y',\theta'+k). \quad (2.11)$$

Plugging this identity in (2.10), we see that instead of integrating over the cusp Z, we can integrate over the half space; this way, we prove that $C(\mathrm{Op}(\sigma),\tau)$ is less than

$$\left[\sup_x \int_{x'\in\mathbb{H}^{d+1}} \left(\frac{y'}{y}\right)^{\frac{d+1}{2}+\tau} |K_\sigma^w(x,x')|dx'\right] \left[\sup_{x'} \int_{x\in\mathbb{H}^{d+1}} \left(\frac{y}{y'}\right)^{\frac{d+1}{2}+\tau} |K_\sigma^w(x,x')|dx\right].$$

By linearity of Op, it suffices to consider real symbols, so we assume that σ is real. Then $K_\sigma^w(x,x') = \overline{K_\sigma^w(x',x)}$. By symmetry of the two terms in the above equation, it suffices to prove that for some $\tau \in \mathbb{R}$, the first term is finite.

Since σ is of order strictly less than $-d-1$, it is integrable in the fibers, and we can estimate :

$$|K_\sigma^w(x,x')| \leqslant \frac{1}{(2\pi h)^{d+1}} \int_{\mathbb{R}^{d+1}} d\xi \left|\sigma\left(\frac{x+x'}{2},\xi\right)\right| \leqslant C_\epsilon \frac{1}{h^{d+1}} \frac{1}{(y+y')^{d+1}} q_{d+1+\epsilon,0}(\sigma).$$

With no decay in θ, this is obviously not sufficient to prove boundedness. We observe the following fact (integrating by parts):

$$\frac{\theta-\theta'}{ih\frac{y+y'}{2}}K^w_\sigma = K^w_{X_J\sigma}. \tag{2.12}$$

Using (2.12) $d+1$ times, and the definition of symbols, we get that :

$$|K^w_\sigma(x,x')| \leqslant C\frac{1}{h^{d+1}}\frac{1}{(y+y')^{d+1}}\frac{1}{1+h^{(d+1)\rho}\left|\frac{\theta-\theta'}{h(y+y')}\right|^{d+1}}. \tag{2.13}$$

Now, we integrate (2.13) in the θ' variable. Actually, we translate by θ, and we rescale with $\mu = h^{\rho-1}(\theta'-\theta)/(y+y')$ to get:

$$\int |K^w_\sigma(y,\theta,y',\theta')|d\theta' \leqslant Ch^{-\rho d-1}\frac{1}{y+y'}\int_{\mathbb{R}^d}\mathrm{d}\mu\frac{1}{1+|\mu|^{d+1}} \leqslant \frac{C}{h^{1+\rho d}}\frac{1}{y+y'}.$$

We just have to find τ such that

$$\sup_{y>0}\int_{y'>0}\frac{1}{y+y'}\left(\frac{y'}{y}\right)^{\frac{d+1}{2}+\tau}dy' < \infty.$$

and then the norm of $\mathrm{Op}(\sigma)$ on L^2 will be $\mathcal{O}(h^{-\rho d-1})$ up to some symbol norm factor. Changing variables to $u = y'/y$, the LHS is

$$\sup_{y>0}\int_0^\infty \frac{u^{\frac{d+1}{2}+\tau}}{1+u}du = \int_0^\infty \frac{u^{\frac{d+1}{2}+\tau}}{1+u}du.$$

For $\tau \in]-(d+1)/2-1, -(d+1)/2[$, this is a convergent integral. □

If we wanted an optimal statement in terms of regularity, we could remark here that we only use $d+1$ symbol estimates (differentiating only in the J direction) to obtain this result.

2.1.2.3. Sobolev regularity

Recall that all functionnal spaces are defined in appendix A.2.

We need a parametrix lemma for the composition of a pseudor with a differential operator:

Lemma 2.1.11. *Let $\sigma \in S^n_\rho$, $Q_{1,2}$ be constant-coefficient elements of $\mathcal{V}(Z)$, of order $k_{1,2}$. Then there exists a symbol $\tilde\sigma \in S^{n+k_1+k_2}_\rho$ such that*

$$Q_1\,\mathrm{Op}(\sigma)Q_2 = \mathrm{Op}(\tilde\sigma),$$
$$\tilde\sigma = \sigma \times \sigma^0(Q_1)\sigma^0(Q_2) + \mathcal{O}(h^{1-\rho}\Psi^{n+k_1+k_2-1}_\rho).$$

Additionally, let $Q_{3,4}$ also be constant-coefficient differential operators with order $k_{3,4}$, satisfying the ellipticity condition *that $\sigma^0(Q_3)\sigma^0(Q_4)$ does not vanish. Take $N \in \mathbb{N}$. Then, there exists a symbol $\tilde\sigma_N$ of order $n+k_1+k_2-k_3-k_4$, such that :*

$$Q_1\,\mathrm{Op}(\sigma)Q_2 = Q_3\,\mathrm{Op}(\tilde\sigma_N)Q_4 + \mathcal{O}(h^N\Psi^{-N}_\rho), \tag{2.14}$$
$$\tilde\sigma_N = \sigma\frac{\sigma^0(Q_1)\sigma^0(Q_2)}{\sigma^0(Q_3)\sigma^0(Q_4)} + \mathcal{O}(h^{1-\rho}\Psi^{n+k_1+k_2-k_3-k_4-1}_\rho).$$

Proof. We start with the first part. Proceeding by induction, we see that it is enough to prove the property when $Q_{1,2}$ are constants, or first order differential operators. The case of constants is straightforward. Now, let us assume $Q_1 = hX_\theta$ and $Q_2 = 1$. The kernel of $Q_1 \operatorname{Op}_h(\sigma)$ is

$$\frac{1}{(2\pi h)^{d+1}} \int_{\mathbb{R}^{d+1}} e^{i\langle x-x',\xi\rangle} \left(\frac{y}{y'}\right)^{\frac{d+1}{2}} \left[iyJ\sigma\left(\frac{x+x'}{2},\xi\right) + \frac{h}{2} y\partial_\theta \sigma\left(\frac{x+x'}{2},\xi\right)\right] d\xi.$$

Decomposing $y = (y+y')/2 + (y-y')/2$, integrating by part in the ξ variable when necessary, we get

$$= \frac{1}{(2\pi h)^{d+1}} \int_{\mathbb{R}^{d+1}} e^{i\langle x-x',\xi\rangle} \left(\frac{y}{y'}\right)^{\frac{d+1}{2}} \left[\sigma_\theta\left(\frac{x+x'}{2},\xi\right)\right] d\xi$$

where

$$\begin{aligned} \sigma_\theta &= iyJ\sigma - \frac{h}{2} yJX_Y\sigma + \frac{h}{2} X_\theta\sigma + i\frac{h^2}{4} X_\theta X_Y \sigma, \\ \sigma_\theta &= iyJ\sigma + \mathcal{O}(h^{1-\rho} S^n_\rho). \end{aligned}$$

similarly, for $Q_1 = hX_y$, we get $Q_1 \operatorname{Op}(\sigma) = \operatorname{Op}(\sigma_y)$ with

$$\sigma_y = iyY\sigma + \frac{h}{2}((d+1)\sigma - yYX_Y\sigma) + \frac{h}{2} X_y\sigma + i\frac{h^2}{4} X_Y X_y \sigma.$$

The case when $Q_1 = 1$ and $Q_2 = hX_\theta, hX_y$ will lead to similar computations, and the same conclusion.

Now, we prove the second part of the lemma. We look for a semi-classical expansion for $\tilde\sigma_N$, in the following form : $\tilde\sigma_N = \sum_0^\infty h^{(1-\rho)k}\sigma_k$ with $\sigma_k \in S^{(n+k_1+k_2-k_3-k_4)-k}$. Injecting this formal development in (2.14), we find a linear system of equations on the σ_k's. Actually, identifying powers of h, we see that this system is in lower-triangular form. The diagonal coefficients are all the same, equal to $\frac{\sigma^0(Q_1)\sigma^0(Q_2)}{\sigma^0(Q_3)\sigma^0(Q_4)}$. The ellipticity condition is sufficient to see that this system has a unique solution of the above form.

Our formal series does not converge, so we truncate at order M for some integer $M \gg 1$. The remainder is then $\mathcal{O}(h^{(M+1)(1-\rho)}\Psi_\rho^{n+k_1+k_2-k_3-k_4-M-1})$. This is certainly $\mathcal{O}(h^N\Psi_\rho^{-N})$ for M big enough; we take $\tilde\sigma_N$ to be this truncated series. □

Proposition 2.1.12. *For all $\epsilon > 0$, the elements of $\Psi^{n-d-1-\epsilon}$ are bounded from H^s to H^{s-n} for all s, n real numbers, with norm $\mathcal{O}(h^{-|s|-|n|-\rho d-1})$.*

Proof. Proceeding by interpolation, we only need to prove this result for s, n even integers. Let $\sigma \in S^k_\rho$ with $k < n-d-1$. Then by (A.4)

$$\|\operatorname{Op}_h(\sigma)\|_{H^s\to H^{s-n}} = h^{-|s|-|n|}\|(P+1)^{(s-n)/2}\operatorname{Op}_h(\sigma)(P+1)^{-s/2}\|_{L^2\to L^2}.$$

By lemma 2.1.11, there is a symbol $\tilde\sigma_N \in S^{k-n}_\rho$ such that

$$(P+1)^{(s-n)/2}\operatorname{Op}_h(\sigma)(P+1)^{-s/2} = \operatorname{Op}_h(\tilde\sigma_N) + (P+1)^{-(s-n)^-/2}\left[\mathcal{O}(h^N\Psi_\rho^{-N})\right](P+1)^{-s^+/2}.$$

Now, we only have to apply lemma 2.1.9 to $\tilde\sigma_N$ to conclude since $(P+1)^{-k}$ is bounded on L^2 as soon as $k \geqslant 0$. □

2.1.2.4. Pseudo-locality statements

Before going on to prove pseudo-locality, we need to define what we mean when we say that a family of operators is smoothing.

Definition 2.1.13. *We say that a family of operators $\{A_h\}_{h>0}$ on $L^2(Z)$ is* smoothing *if for every $h > 0$ and $n > 0$, A_h maps H^{-n} to H^n in a continuous fashion. Additionally, we require that the following semi-norms*

$$\|.\|_{n,n} = \sup_{h>0} \|.\|_{H^{-n}\to H^n}, \quad n \in \mathbb{N}.$$

are finite. We refer to the space of smoothing operators as $\Psi^{-\infty}$. The semi-norms give a topology of Fréchet space to $\Psi^{-\infty}$.

A family $\{A_h\}_{h>0}$ is said to be asymptotically *smoothing if for every $n > 0$, there is a $h_n > 0$ such that for every $0 < h < h_n$, A_h is uniformly bounded from H^{-n} to H^n. This space is also endowed with semi-norms*

$$\|.\|_{n,n,k} = \sup_{0<h<1/k} \|.\|_{H^{-n}\to H^n} \quad n, k \in \mathbb{N}.$$

Finally, we say that a family of operators is (asymptotically) negligible if it is $\mathcal{O}(h^\infty)$ in the space of (asymptotically) smoothing operators. The space of negligible operators is denoted $\mathcal{O}(h^\infty)\Psi^{-\infty}$. We let $\Psi_\rho = \mathcal{O}(h^\infty)\Psi^{-\infty} \cup_{n\in\mathbb{R}} \Psi^n_\rho$.

We deduce of proposition 2.1.12

Corollary 2.1.14. *The composition of a negligible (resp. asymptotically negligible) operator with a pseudor is still a negligible (resp. asymptotically negligible) operator.*

Notation 2.1. *We denote by $S(\mathbb{R}, \alpha)$ the class of symbols on $\mathbb{R}$ of order α, meaning that $\eta \in S(\mathbb{R}, \alpha)$ when for all $k \geqslant 0$, there is a constant $C_k > 0$ with*

$$\eta^{(k)}(u) \leqslant C_k \langle u\rangle^{\alpha-k}.$$

Let K be the kernel of some operator A on $C^\infty_c(Z)$. Let $\eta \in S(\mathbb{R}, \alpha)$. Then define A_η to be the operator with kernel

$$K_\eta(x, x') = K(x, x')\eta\left(\frac{y'}{y} - \frac{y}{y'}\right).$$

Proposition 2.1.15. *Let $\eta \in S(\mathbb{R}, \alpha)$ vanish near 0 with $\alpha \leqslant 0$. Let $\sigma \in S_\rho$. Then $\mathrm{Op}(\sigma)_\eta$ is negligible.*

Proof. We first give bounds on L^2. Recall that K^w_σ is the kernel of the operator in (2.7). Similarly to (2.12), we have:

$$\frac{y - y'}{ih\frac{y+y'}{2}} K^w_\sigma = K^w_{X_Y\sigma}. \tag{2.15}$$

from this, we deduce that for all $N \in \mathbb{N}$,

$$\mathrm{Op}(\sigma)_\eta = (ih/2)^N \,\mathrm{Op}((X_Y)^N\sigma)_{\eta_N},$$

where

$$\eta_N\left(t-\frac{1}{t}\right)=\left(\frac{1+t}{1-t}\right)^N\eta\left(t-\frac{1}{t}\right),$$

so that $\eta_N \in S(\mathbb{R},\alpha)$ (with the same α). For N big enough, $X_Y^N\sigma h^{-\rho N} \in S_\rho^{-d-2}$. From lemma 2.1.9, we thus determine that for all $N \geqslant 0$

$$\|\operatorname{Op}_h(\sigma)_\eta\|_{L^2\to L^2}=\mathcal{O}(h^{N(1-\rho)-\rho d-1}),$$

where the constant depends on $\|\eta_N\|_\infty$ — which are finite since $\alpha \leqslant 0$.

Now, up to some fixed power of h, the $H^{-2N}\to H^{2N}$ norm is bounded by

$$\|(P+1)^N\operatorname{Op}_h(\sigma)_\eta(P+1)^N\|_{L^2\to L^2}. \tag{2.16}$$

Observe that composition with X_θ commutes with the $A\to A_\eta$ operation. Further,

$$X_y\left[\eta\left(\frac{y}{y'}-\frac{y'}{y}\right)\right]=\left[\frac{y}{y'}+\frac{y'}{y}\right]\eta'\left(\frac{y}{y'}-\frac{y'}{y}\right)=\eta^*\left(y-\frac{1}{y}\right)\text{ with }\eta^*\in S(\mathbb{R},\alpha).$$

Combining this with (2.15), we deduce that if we expand both $(P+1)^N$'s in (2.16), we will get a finite sum of $\operatorname{Op}(\sigma^*)_{\eta^*}$, with σ^* in S_ρ, and η^* in $S(\mathbb{R},\alpha)$ still vanishing near 0. We can apply the first part of our proof to conclude. □

Remark 2.1. *Actually, if we take $\eta(u)=\tilde{\eta}(u/h^{\rho'})$, with $\tilde{\eta}\in S(\mathbb{R},0)$, and go through the above proof, we see that it works as long as $\rho' < 1-\rho$. We deduce that the kernel of $\operatorname{Op}_h(\sigma)$ is essentially supported at distance $h^{1-\rho}$ of $\{y=y'\}$.*

2.1.3. Stationary Phase

Now that we proved that off-diagonal terms in the kernel of our pseudors give rise to negligible operators, it is legitimate to cutoff the kernels and keep only the part supported near the diagonal. While proving composition formulae, or when changing quantizations, this will produce in the equations expressions of the type

$$\sigma_1(x_0,\xi_0)\times\sigma_2(x_1,\xi_1)\times\chi(x_0,x_1)$$

where $\chi(x_1,x_2)$ is a function of y_1/y_2 supported near 1. This motivates the introduction of

Definition 2.1.16. *For $1>\epsilon>0$, let $\Omega_{k,\epsilon}$ be the subset of $(T^*Z)^{k+1}$:*

$$\Omega_{k,\epsilon}=\left\{(x_0,\xi_0;x_1,\xi_1,\ldots,x_k,\xi_k)\in(T^*Z)^{k+1}\quad\middle|\quad\forall i,\epsilon\leqslant\left|\frac{y_i}{y_0}\right|\leqslant 1/\epsilon\right\}.$$

Let σ be some smooth function on $(T^*Z)^{k+1}\times[0,h_0[$, supported in some $\Omega_{k,\epsilon}$. We say that σ is a (k,ρ)-symbol if it is a symbol w.r.t the weights

$$\{\langle y_0\xi_0\rangle^{\beta_0}\ldots\langle y_0\xi_k\rangle^{\beta_k}|(\beta_0,\ldots,\beta_k)\in\mathbb{R}^{k+1}\}$$

and the vector fields $X_{x,i} = y_0\partial_{x_i}$ and $X_{\xi,i} = 1/y_0\partial_{\xi_i}$, losing a constant $h^{-\rho}$ when differentiating. By this we mean that there is $\beta \in \mathbb{R}^{k+1}$ such that whenever α is a finite sequence of indices $\alpha_j \in \{(x_i),(\xi_i)|i = 0\dots k\}$, if n_i is the number of (ξ_i) in the sequence,

$$|X_\alpha \sigma| \leqslant C_\alpha \langle y_0\xi_0\rangle^{\beta_0-n_0}\dots\langle y_0\xi_k\rangle^{\beta_k-n_k} \text{ for some constant } C_\alpha > 0. \quad (2.17)$$

In particular, a $(0,\rho)$-symbol is just a symbol in S_ρ. The semi-norms defined in (2.17) give a topology to the space of (k,ρ)-symbols.

For σ a (k,ρ)-symbol, we define the following function on T^*Z:

$$T_k\sigma : (x,\xi,h) \mapsto (2\pi h)^{-k(d+1)}\int_{i=1\dots k} e^{\frac{i}{h}[\sum\langle x_i - x,\xi_i-\xi\rangle]}\sigma(x,\xi;(x_1,\xi_1),\dots,(x_k,\xi_k))dx_i d\xi_i,$$

where the integration has been taken over the universal cover $T^*(\mathbb{H}^{d+1})^k$.

Remark 2.2. *$T_k\sigma$ is well defined. Indeed, if we first perform the integration in the ξ_i variables, we obtain Fourier transforms of symbols. Those are distributions whose singular support is reduced to $\{0\}$ and are decreasing faster than any power at infinity. Using compact support in $y_{1,\dots,k}$ — depending on y_0 — we see that such distributions can be integrated against 1.*

Recall the notation

$$\nabla_x.\nabla_\xi = \partial_y\partial_Y + \sum_{i=1\dots d}\partial_{\theta_i}\partial_{J_i}$$

Proposition 2.1.17. *Let σ be some (k,ρ)-symbol, of order β. Then $T_k\sigma$ is in $S_\rho^{|\beta|}(Z)$ and we have the following expansion :*

$$T_k\sigma(x,\xi,h) \sim \sum_{\alpha\in\mathbb{N}^k}\frac{(ih)^{|\alpha|}}{\alpha!}\left[\prod_1^k(\nabla_{x_i}.\nabla_{\xi_i})^{\alpha_i}\right]\sigma(x,\xi;x_1,\xi_1,\dots,x_k,\xi_k)\big|_{(x_i,\xi_i)=(x,\xi)}.$$

In addition, T_k is continuous from the space of (k,ρ)-symbols to S_ρ.

Remark 2.3. *All the terms in the expansion are in the right symbol class : if σ is a (k,ρ)-symbol of order β, the terms with coefficient $h^{|\alpha|}$ are in finite number, and are symbols in $S_\rho^{|\beta|-|\alpha|}$.*

We will only use this proposition for $k = 1$ and $k = 2$

Proof. We prove the result by induction on k. First, if $k = 0$, this is obvious, since $T_0\sigma = \sigma$. Now, if we assume it is true for k, let σ be some $(k+1,\rho)$ symbol. Then, we can consider that σ is a (k,ρ) symbol in its k first coordinates, with the last coordinates as parameters. Applying T_k on those first coordinates, we obtain that $T_k\sigma$ is a $(1,\rho)$ symbol by the induction hypothesis (here, the continuity of $T_k\sigma$ is important). Then, we remark that $T_{k+1}\sigma = T_1T_k\sigma$.

Hence, proving the announced property for T_1 is sufficient. Assume for now that :

Lemma 2.1.18. *If σ is a $(1,\rho)$ symbol of order (k_0,k_1) with $k_1 < 0$, for some constant*

$$|T_1\sigma(x,\xi,h)| \leqslant C\langle y\xi\rangle^{k_0+k_1},$$

with C controlled by a finite number of symbol norms.

Let σ be some $(1,\rho)$ symbol of order (k_0, k_1). Changing variables $(v, V) = (x_1, \xi_1) - (x, \xi)$ in $T_1\sigma$,

$$T_1\sigma : (x, \xi, h) \mapsto (2\pi h)^{-(d+1)} \int e^{\frac{i}{h}\langle v, V\rangle} \sigma(x, \xi; (x, \xi) + (v, V)) dv dV. \tag{2.18}$$

Hence, the following identities hold (integrating by parts):

$$\begin{aligned} X_y T_1\sigma &= T_1(y_0 \partial_{y_0}\sigma) + T_1(y_0 \partial_{y_1}\sigma) \\ X_\theta T_1\sigma &= T_1(y_0 \partial_{\theta_0}\sigma) + T_1(y_0 \partial_{\theta_1}\sigma) \text{ where } \theta_i \in \mathbb{R}^d \text{ is the } \theta \text{ coordinate for } x_i \\ X_Y T_1\sigma &= T_1\left(\frac{1}{y_0}(\partial_{Y_0}\sigma + \partial_{Y_1}\sigma)\right) \\ X_J T_1\sigma &= T_1\left(\frac{1}{y_0}(\partial_{J_0}\sigma + \partial_{J_1}\sigma)\right). \end{aligned}$$

We deduce then from lemma 2.1.18 that $T_1\sigma$ is in S_ρ with the correct order whenever $k_1 < 0$, and depends continuously on σ. Now, in the general case, we apply Taylor's formula to the V variable :

$$\sigma(x, \xi; x + v, \xi + V) = \\ \sum_{s=0}^{n} \frac{1}{s!} d_{\xi_1}^s \sigma(x, \xi; (x + v, \xi)).V^{\otimes s} + \int_0^1 \frac{(1-t)^n}{n!} (d_{\xi_1}^{n+1}\sigma)(x, \xi; x + v, \xi + tV).V^{\otimes n+1} dt.$$

Plugging this in the formula for $T_1\sigma$, integrating by parts in the v variable, we obtain a sum of symbols

$$\sum_{s=0}^{n} \frac{(ih)^s}{s!} (\nabla_{x_1}.\nabla_{\xi_1})^s \left(\sigma(x, \xi; x_1, \xi_1)\right)_{|(x_1,\xi_1)=(x,\xi)}, \tag{2.19}$$

which depends continuously on σ. We also have a remainder term

$$(2\pi h)^{-(d+1)} (ih)^{n+1} \int_0^1 \frac{(1-t)^n}{n!} \int e^{\frac{i}{h}\langle v, V\rangle} \left[(\nabla_{x_1}.\nabla_{\xi_1})^{n+1}\sigma\right](x, \xi; x + v, \xi + tV) dv dV dt$$

Actually, after rescaling V by a factor t, this remainder term is seen to be

$$(ih)^{n+1} \int_0^1 \frac{(1-t)^n}{n!} T_1\sigma^*(x, \xi, th) dt$$

where σ^* is of order $(k_0, k_1 - n - 1)$ and depends continuously on σ (we only took a finite number of derivatives). If we take $n \geqslant k_1$, we already know that $T_1\sigma^*$ is a symbol depending continuously on σ^*, so that the remainder is $\mathcal{O}(h^{(n+1)(1-2\rho)})$ is S_ρ^n, with constants depending on a finite number of derivatives of σ. Together with remark 2.3, this is enough to conclude. □

Now, let us prove lemma 2.1.18.

Proof. We rescale variable V in (2.18) to $W = V/h$, absorbing the h^{-d-1} constant. Let $\chi \in C_c^\infty(\mathbb{R}^{d+1})$ equal 1 near 0, and break the integral into two parts with $1 = (\chi + (1 -$

$\chi))(h^\rho yW)$. In the part with $1-\chi$, we can also introduce $1 = (h^\rho yW)^{2N}/(h^\rho yW)^{2N}$ for some N big enough, and get

$$\int e^{i\langle v,W\rangle}\left[\chi(h^\rho yW) + (1-\chi(h^\rho yW))\frac{(h^\rho yW)^{2N}}{(h^\rho yW)^{2N}}\right]\sigma(x,\xi;x+v,\xi+hW)dvdW.$$

If we integrate the second term by parts in the v variable $2N$ times, we get rid of the $(h^\rho yW)^{2N}$ on top. We see that for both terms we obtain an expression of the form

$$\int e^{i\langle v,W\rangle}\psi(h^\rho yW)\sigma^*(x,\xi;x+v,\xi+hW)dvdW$$

where either $(\sigma^*,\psi) = (\sigma,\chi)$ or $(\sigma^*,\psi) = (h^{2\rho N}y^{2N}(\partial_{y_1}^2+\partial_{\theta_1}^2)^N\sigma, (1-\chi(x))/x^{2N})$. In both cases, σ^* has the same properties as σ (including support, bounds, and order), and ψ is some symbol on $\mathbb{R}^{d+1}$ in the usual sense, of order $-2N$. We apply the same trick in the v variable now, introducing

$$1 = \chi(h^{-\rho}v/y) + (1-\chi)(h^{-\rho}v/y)\frac{(h^{-\rho}v/y)^{2N}}{(h^{-\rho}v/y)^{2N}}$$

and integrating by parts in the W variable for the second term. When differentiating ψ, the powers of h compensate; when differentiating σ^*, we gain a positive power $h^{1-2\rho}$ — here $\rho < 1/2$ is important. In the end, we get new expressions of the form

$$\int e^{i\langle v,W\rangle}\psi(h^\rho yW)\tilde{\psi}(h^{-\rho}v/y)\sigma^*(x,\xi;x+v,\xi+hW)dvdW$$

where σ^* — not the same as before — still has the same properties as σ, and $\psi,\tilde{\psi}$ are some symbols on $\mathbb{R}^{d+1}$ in the usual sense, of order $-2N$. We can take the L^1 norm of the integrand, and it is bounded by :

$$C\int_{\mathbb{R}^{2(d+1)}}\langle h^\rho yW\rangle^{-2N}\langle h^{-\rho}v/y\rangle^{-2N}\langle y(\xi+hW)\rangle^{k_1}\langle y\xi\rangle^{k_0}dvdW.$$

rescaling both v and W, this is bounded by

$$\begin{aligned}&C\langle y\xi\rangle^{k_0}\int_{\mathbb{R}^{2(d+1)}}\langle v\rangle^{-2N}\langle W\rangle^{-2N}\langle y\xi+h^{1-\rho}W\rangle^{k_1}dvdW\\ &\leqslant C\langle y\xi\rangle^{k_0}\int_{\mathbb{R}^{d+1}}\langle W\rangle^{-2N}\langle y\xi+h^{1-\rho}W\rangle^{k_1}dW \quad \text{for } N>d.\end{aligned}$$

We break the integral into two parts : $\{|W| > \varepsilon|y\xi|\}$ and $\{|W| \leqslant \varepsilon|y\xi|\}$ for some $\epsilon > 0$. Since $k_1 < 0$, $\langle y\xi + hW\rangle^{k_1} \leqslant 1$, and the first part is bounded by

$$C\langle y\xi\rangle^{k_0}\int_{|W|>\varepsilon|y\xi|}\langle W\rangle^{-2N} = \mathcal{O}(\langle y\xi\rangle^{k_0+d-N+1}) = \mathcal{O}(\langle y\xi\rangle^{k_0+k_1}) \text{ when } N \geqslant k_1+d+1.$$

The second part is bounded by

$$C\langle y\xi\rangle^{k_0+k_1}\times\int_{\mathbb{R}^{d+1}}\langle W\rangle^{-2N} = \mathcal{O}(\langle y\xi\rangle^{k_0+k_1}).$$

□

2.1.4. Symbolic calculus consequences

We start this section by proving that the class of operator Ψ_ρ is stable by composition.

Proposition 2.1.19. *Let $\sigma_1 \in S^m_\rho(Z)$ and $\sigma_2 \in S^n_\rho(Z)$. Then, there is a symbol $\sigma_1 \# \sigma_2 \in S^{m+n}_\rho(Z)$ and a negligible family of operators $R_h \in \mathcal{O}(h^\infty)\Psi^{-\infty}$ such that*

$$\mathrm{Op}(\sigma_1)\,\mathrm{Op}(\sigma_2) = \mathrm{Op}(\sigma_1\#\sigma_2) + R_h$$

where

$$\sigma_1\#\sigma_2(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^2} \frac{(-1)^{\alpha_2}(ih)^{|\alpha|}}{2^{|\alpha|}\alpha!}(\nabla_{x_1}.\nabla_{\xi_2})^{\alpha_1}(\nabla_{x_2}.\nabla_{\xi_1})^{\alpha_2}\sigma_1(x_1,\xi_1)\sigma_2(x_2,\xi_2)_{|x_1=x_2=x,\xi_1=\xi_2=\xi}. \tag{2.20}$$

Proof. First, we choose a truncation $\eta \in C^\infty_c(\mathbb{R})$ equal to 1 around the origin. Then $1-\eta$ is a symbol in $S(\mathbb{R},0)$ vanishing around 0, so we can apply proposition 2.1.15, and replace $\mathrm{Op}(\sigma_1)$ and $\mathrm{Op}(\sigma_2)$ by respectively $\mathrm{Op}(\sigma_1)_\eta$ and $\mathrm{Op}(\sigma_2)_\eta$. Recalling corollary 2.1.14, there exists $R_h \in \mathcal{O}(h^\infty)\Psi^{-\infty}$ such that

$$\mathrm{Op}(\sigma_1)\,\mathrm{Op}(\sigma_2) = \mathrm{Op}(\sigma_1)_\eta\,\mathrm{Op}(\sigma_2)_\eta + R_h.$$

If K^w_σ is the kernel of $\mathbf{Op}^w_h(\sigma)$ on $\mathbb{H}^{d+1}$ as in (2.7), we have

$$\sigma(x,\xi) = \int e^{\frac{i}{h}\langle u,\xi\rangle} K^w_\sigma\left(x-\frac{u}{2}, x+\frac{u}{2}\right) du.$$

Since both $\mathbf{Op}^w_h(\sigma_1)_\eta$ and $\mathbf{Op}^w_h(\sigma_2)_\eta$ act on $\mathbb{H}^{d+1}$, the product also, and its kernel on $\mathbb{H}^{d+1}$ is

$$K^w(x,\tilde{x}) = \int K^w_{\sigma_1}(x,x')K^w_{\sigma_2}(x',\tilde{x})\eta\left(\frac{y_{x'}}{y_x}-\frac{y_x}{y_{x'}}\right)\eta\left(\frac{y_{\tilde{x}}}{y_{x'}}-\frac{y_{x'}}{y_{\tilde{x}}}\right)dx'$$

Hence, the solution to our problem is (formally) the function $\sigma_1\#\sigma_2$ defined by

$$\sigma_1\#\sigma_2(x,\xi) = h^{-2d-2}\int e^{\frac{i}{h}\phi}\tilde{\sigma}(u,x',\xi_1,\xi_2)\chi\left(\frac{y+y_u/2}{y'},\frac{y-y_u/2}{y'}\right)dud\xi_1dx'd\xi_2$$

integrating over $(T^*\mathbb{R}^{d+1})^2$, where

$$\phi = \langle u,\xi\rangle + \langle x-u/2-x',\xi_1\rangle + \langle x'-x-u/2,\xi_2\rangle$$
$$\tilde{\sigma}(u,x',\xi_1,\xi_2) = \sigma_1\left(\frac{x-u/2+x'}{2},\xi_1\right)\sigma_2\left(\frac{x+u/2+x'}{2},\xi_2\right)$$

and χ is a smooth function on $\mathbb{R}^2$ supported in a rectangle

$$\{(\tau_1,\tau_2)\in\mathbb{R}^2 | 0<\epsilon\leqslant\tau_1\leqslant 1/\epsilon \quad 0<\epsilon\leqslant\tau_2\leqslant 1/\epsilon\}$$

After a change of variables, we will be able to use our stationary phase lemma. Let

$$x_1 = \frac{1}{2}(x+u/2+x') \quad x_2 = \frac{1}{2}(x-u/2+x').$$

we get to write c in the suitable fashion

$$\sigma_1 \# \sigma_2(x,\xi) = \left(\frac{2}{h}\right)^{-2d-2} \int e^{\frac{2i}{h}(\langle x-x_1,\xi-\xi_1\rangle + \langle x-x_2,\xi-\xi_2\rangle)} \tilde{\sigma}' \chi_2 = T_2(\tilde{\sigma}'\chi_2)(x,\xi,h/2),$$

where

$$\begin{aligned} \tilde{\sigma}' &= \sigma_1(x_2,\xi_1)\,\sigma_2(x_1,\xi_2), \\ \chi_2 &= \chi\left(\frac{y_1-y_2+y}{y_1+y_2-y}, \frac{y_2-y_1+y}{y_1+y_2-y}\right). \end{aligned}$$

An elementary computation shows that χ_2 is supported in some $\{\epsilon' y \leqslant y_{1,2} \leqslant y/\epsilon'\}$; hence, it is a smooth function of $y_{1,2}/y$ and it will have a good behavior w.r.t vector fields $y\partial_{y_{012}}$. Function $\tilde{\sigma}'\chi_2$ is supported in $\Omega_{2,\epsilon'}$. In addition, since the weights $(\langle y_i \xi_j \rangle)_{i=1,2}$ are equivalent to $\langle y_0 \xi_j \rangle$ in $\Omega_{2,\epsilon}$, $\tilde{\sigma}'$ satisfies the desired estimates in that region, and $\tilde{\sigma}'\chi_2$ is a $(2,\rho)$-symbol. From proposition 2.1.17, we conclude directly that $\sigma_1 \# \sigma_2$ is in $S^{m+n}_\rho(Z)$. □

Proposition 2.1.20. *The adjoint of* $\mathrm{Op}_h(\sigma)$ *for the* L^2 *inner product is* $\mathrm{Op}_h(\overline{\sigma})$*, so that real symbols yield essentially self-adjoint operators, which is a key feature of the Weyl quantization.*

Proof. Recall that with $\mathcal{L} : f \in L^2(Z) \to y^{-(d+1)/2} f \in L^2(dyd\theta)$,

$$\mathrm{Op}(\sigma) = \mathcal{L}^* \mathbf{Op}^w_h(\sigma) \mathcal{L}.$$

Since the usual Weyl quantization on $\mathbb{R}^n$ has the property we announce for Op, we deduce the first part of the proposition: $\mathrm{Op}_h(\sigma)^* = \mathrm{Op}_h(\overline{\sigma})$.

Now, we use proposition 8.5 in appendix A in Taylor [Tay11] to show that $\mathrm{Op}(\sigma)$ is self adjoint. It suffices to prove that when σ is real, $\mathrm{Op}_h(\sigma) \pm i$ is surjective. Since σ is real, $\sigma \pm i$ never vanishes, and we can find a symbol $\sigma^\pm_N$ such that

$$\mathrm{Op}(\sigma \pm i)\,\mathrm{Op}(\sigma^\pm_N) = 1 + \mathcal{O}(h^N \Psi^{-N}).$$

for h small enough, the operator on the RHS is invertible. In particular it is surjective, and so is $\mathrm{Op}_h(\sigma \pm i)$. □

Proposition 2.1.21. *Let* σ_1 *and* σ_2 *be in* $S_\rho(Z)$*. Then, with* $R_h \in \mathcal{O}(h^\infty)\Psi^{-\infty}$*,*

$$[\mathrm{Op}(\sigma_1), \mathrm{Op}(\sigma_2)] = \mathrm{Op}(\sigma_3) + R_h$$

where σ_3 *is a semi-classical symbol with an asymptotic expansion with only odd powers of* h*, such that :*

$$\sigma_3(x,\xi) = \frac{h}{i}\{\sigma_1,\sigma_2\} + \mathcal{O}(h^{3(1-2\rho)} S^{n+m-3}_\rho)$$

where $\{.,.\}$ *is the Poisson bracket defined with the symplectic form* $d\xi \wedge dx$ *:*

$$\{f,g\} = \nabla_\xi f . \nabla_x g - \nabla_x f . \nabla_\xi g.$$

Proof. Since in the asymptotic expansion (2.20) the terms in odd powers of h are symmetric in σ_1 and σ_2, this other key feature of Weyl quantization is now simple to observe. □

Proposition 2.1.22. *Let $\sigma \in S^0_\rho(Z)$. Then $\mathrm{Op}(\sigma)$ is bounded on L^2, with norm $\|\sigma\|_\infty + o_{h\to 0}(1)$.*

Proof. We have all the ingredients to make the classical proof work. Consider

$$\|\mathrm{Op}_h(\sigma)\|^2_{L^2} = \|\mathrm{Op}_h(\sigma)^* \mathrm{Op}_h(\sigma)\|_{L^2}$$
$$\mathrm{Op}_h(\sigma)^* \mathrm{Op}_h(\sigma) = \mathrm{Op}_h(|\sigma|^2) + \mathcal{O}(hS^{n-1}_\rho) \text{ for } \sigma \in S^n_\rho(Z)$$

When σ has negative order, $|\sigma|^2$ has a more negative order. Since Ψ^{-d-2} operators are bounded with norm $O(h^{-\rho d-1})$, one can prove by induction that for any ϵ, ϵ', $\Psi^{-\epsilon}$ operators are bounded on L^2 whenever $\epsilon > 0$, with norm $\mathcal{O}(h^{-\epsilon'})$. Now, take $\sigma \in S^0_\rho(Z)$. Let $M = \|\sigma\|^2_\infty$. We just have to prove that $M - \mathrm{Op}_h(\sigma)^* \mathrm{Op}_h(\sigma) + o(1)$ is a positive operator. Take $\tau > 0$. Consider

$$M + \tau - \mathrm{Op}_h(\sigma)^* \mathrm{Op}_h(\sigma) = \mathrm{Op}_h(M + \tau - |\sigma|^2) + O(h^{1-\rho} S^{-1}_\rho).$$

But $M + \tau + |\sigma|^2 > \tau$ so $\sigma' = \sqrt{M + \tau + |\sigma|^2}$ is in $S^0_\rho(Z)$ and real, so that $\mathrm{Op}(\sigma')$ is self-adjoint, and

$$M + \tau + \mathrm{Op}_h(\sigma)^* \mathrm{Op}_h(\sigma) = \mathrm{Op}_h(\sigma')^2 + \mathcal{O}(h^{1-\rho} S^{-1}_\rho) \geqslant -C.h^{1-\rho-\epsilon'} \text{ for any } \epsilon' \text{ given.}$$

We deduce that $M - \mathrm{Op}_h(\sigma)^* \mathrm{Op}_h(\sigma) \geqslant -Co(1)$. □

Proposition 2.1.23. *Let $f \in S(\mathbb{R}, n)$. With $P = -h^2\Delta/2$, we define $f(P)$ by the spectral theorem. Recall that p is the symbol of P. Then there is a symbol σ such that*

$$\sigma = f \circ p + \mathcal{O}(hS^{n-1})$$

and

$$f(P) = \mathrm{Op}(\sigma) + R$$

where R is asymptotically *negligible and commutes with ∂_θ.*

Proof. If f has positive order n, consider

$$f(P) = (P + i)^{n+1} \frac{f}{(x + i)^{n+1}}(P).$$

Since $(P + i)^{n+1}$ is a pseudor, we only need to consider cases when f has negative order. Following the method in lemma 2.1.11, we get symbols $q_N(z)$ and $r_N(z)$ such that

$$(P + z)\,\mathrm{Op}(q_N(z)) = \mathbb{1} + \mathrm{Op}(h^N r_N(z)).$$

What is more, the symbol norms of q_N and r_N are bounded by a power $|\Im z|^{-L_N}$ with $L_N \to \infty$ when $N \to \infty$. This establishes that $(P + z)^{-1}$ is a pseudor up to an asymptotically negligible remainder, for fixed z. Now, using a quasi-analytic extension of f as in p.358 in [Zwo12] (theorem 14.8 therein), and the bounds on q_N and r_N, we see that the same can be said about $f(P)$.

To conclude, observe that $\mathrm{Op}(\sigma)$ commutes with ∂_θ whenever σ does not depend on θ, which is the case for q_N and r_N. □

To prove a trace formula, it is convenient to be able to change quantizations.

Lemma 2.1.24. *On $\mathbb{R}^{d+1}$, we can define a family of quantization as usual by*

$$\mathbf{Op}_h^t(\sigma)f(x) = \frac{1}{(2\pi h)^{d+1}} \int e^{i\langle x-x',\xi\rangle/h} \sigma\left(tx + (1-t)x', \xi\right) f(x') d\xi dx'$$

and then define $\mathrm{Op}_h^t(\sigma) := \mathcal{L}^* \mathbf{Op}_h^t(\sigma)\mathcal{L}$, *for* $\sigma \in S_\rho(Z)$ *— so that* $\mathrm{Op}^{1/2} = \mathrm{Op}$.

If $\sigma \in S_\rho^n$, *there is a family* σ_t *of symbols so that for all* $t \in [0,1]$,

$$\mathrm{Op}^t(\sigma_t) = \mathrm{Op}(\sigma) + \mathcal{O}(h^\infty \Psi^{-\infty}).$$

What is more

$$\sigma_t = \sigma + \mathcal{O}(h^{1-\rho} S_\rho^{n-1}).$$

Proof. First,the results we gave above for $\mathrm{Op} = \mathrm{Op}^{1/2}$ have an equivalent for Op^t for all $t \in [0,1]$. The proofs are similar. In particular, they obey the same bounds on Sobolev spaces.

Following the scheme of proof of the composition lemma, we truncate the kernel of $\mathrm{Op}^{1/2}$ around $y = y'$ with a η compactly supported, and we want to solve

$$\mathrm{Op}^t(\sigma_t)_\eta = \mathrm{Op}(\sigma)_\eta.$$

If K^t is the kernel of $\mathbf{Op}_h^t(\sigma_t)$, we have

$$\sigma_t(x,\xi) = \int e^{i\langle u,\xi\rangle/h} K^t(x + (t-1)u, x + tu) du.$$

so it is legitimate to consider

$$\begin{aligned} \sigma_t'(x,\xi) &:= (2\pi h)^{-d-1} \int e^{i\langle u,\xi-\xi'\rangle/h} \sigma\left(x + (t-1/2)u, \xi'\right) \chi\left(\frac{y + ty_u}{y + (t-1)y_u}\right) du d\xi' \\ &= T_1(\sigma_t'')(x,\xi,h) \end{aligned}$$

with

$$\sigma_t''(x,\xi;x_1,\xi_1) = \sigma\left((1/2+t)x + (1/2-t)x_1, \xi_1\right) \chi\left(\frac{y(1+t) - ty_1}{ty + y_1(1-t)}\right)$$

and

$$\chi(x) = \eta(x - 1/x).$$

One can check that σ_t'' is a $(1,\rho)$-symbol. We deduce then from proposition 2.1.17 that

$$\mathrm{Op}(\sigma) = \mathrm{Op}^t(\sigma_t') + \mathcal{O}(h^\infty \Psi^{-\infty}).$$

and

$$\sigma_t' = \sigma + \mathcal{O}(h^{1-\rho} S_\rho^{n-1}) \in S_\rho^n.$$

□

Before we turn to a trace formula, observe that when one imposes Dirichlet conditions at $y = y_0$ and considers the Laplacian on $L^2(Z, \{y > y_0\})$, one finds that it has continuous spectrum $[d^2/4, +\infty)$. We cannot expect our operators to be trace class, if they are not even compact. This is why we introduce the following.

Let Π^* be the orthogonal projection in $L^2(Z)$ on the non-zero Fourier modes in the θ direction. Also let Λ' be the dual lattice to Λ and $\Lambda'^* = \Lambda' \backslash \{0\}$.

Lemma 2.1.25. *Let $\epsilon > 0$ and $a > 0$. Let $\chi \in \mathscr{C}_b^\infty(Z)$ be supported in $\{y > a\}$. When $\sigma \in S_\rho^{-(d+1)/2-\epsilon}$ is supported in $\{y > a\}$, both $\mathrm{Op}^1(\sigma)\Pi^*$ and $\Pi^* \mathrm{Op}^0(\sigma)$ are Hilbert-Schmidt. As a consequence, for any $A \in \Psi_\rho^{-(d+1)/2-\epsilon}$, both $\Pi^*\chi A \chi$ and $\chi A \chi \Pi^*$ are Hilbert-Schmidt — this is also true if A is asymptotically negligible.*

Proof. The Hilbert-Schmidt (HS) norm (on $L^2(Z)$) of an operator A with kernel K w.r.t to the Lebesgue measure on the cylinder is

$$\|A\|_{HS}^2 = \int_{Z \times Z} |K(x, x')|^2 \left(\frac{y'}{y}\right)^{d+1} dy d\theta dy' d\theta'.$$

Recall the Poisson formula (the covolume of Λ is 1)

$$\sum_{\varpi \in \Lambda} e^{i\langle \varpi, W\rangle} = \sum_{W_i \in \Lambda'} \delta_{W_i}(W).$$

Using (2.11), we deduce that the kernel of $\mathrm{Op}^1(\sigma)$ is

$$K_\sigma^1(x, x') = (2\pi)^{-d-1} \left(\frac{y}{y'}\right)^{\frac{d+1}{2}} \sum_{J \in \Lambda'} \int e^{i\langle x - x', \xi\rangle} \sigma(x, h\xi)\, dY, \text{ where } \xi = (Y, J).$$

Seing this as a Fourier transform in the x' variable, by Parseval :

$$\| \mathrm{Op}_1(\sigma_1)\Pi^* \|_{HS}^2 = \frac{1}{(2\pi)^d} \sum_{J \in \Lambda'^*} \int_{Z \times \mathbb{R}} |\sigma_1(y, \theta, hY, hJ)|^2 dy d\theta dY$$

Since $\sigma \in S_\rho^{-(d+1)/2-\epsilon}$ is supported in $\{y \geqslant a\}$, this is less than

$$C \sum_{J \in \Lambda'^*} \int_a^\infty \int_{\mathbb{R}} \langle h^2 y^2 (Y^2 + J^2)\rangle^{-d-1-2\epsilon} dY dy \leqslant C_N \sum_{J \in \Lambda'^*} \int_a^\infty \frac{dy}{hy(1 + (hy|J|)^2)^{d/2+\epsilon}}.$$

After some change of variables, this is seen to be finite for fixed h. Observe that if we did not remove $J = 0$, it would not be the case; there would be a logarithmic divergence. Hence $\mathrm{Op}^1(\sigma)\Pi^*$ is HS. Taking adjoints, we see that that $\Pi^* \mathrm{Op}^0(\overline{\sigma})$ is also HS.

Then, we write for some $N > 0$ big enough.

$$(P + 1)^{-(d+1)/4-\epsilon} = \mathrm{Op}^1(q_N^1)(1 + \mathrm{Op}^1(h^N r_N^1))^{-1} = (1 + \mathrm{Op}^0(h^N r_N^0))^{-1} \mathrm{Op}^0(q_N^0)$$

We deduce that both

$$\Pi^* \chi (P + 1)^{-(d+1)/4-\epsilon} \text{ and } \Pi^* (P + 1)^{-(d+1)/4-\epsilon} \chi$$

are HS, being the product of a HS and a bounded operator.

Take R asymptotically negligible, whose kernel is supported in $\{y, y' > a\}$. Then for any N, $(P+1)^N R$ is bounded on L^2, so that we can write $\Pi^* R = \Pi^* \tilde{\chi}(P+1)^{-N}\tilde{R}$ where $\tilde{\chi}$ is supported in some $\{y > a' > 0\}$, equal to 1 on the support of R, and $\tilde{R}$ is a bounded operator for h small enough. We deduce that $\Pi^* R$ is HS for h small enough, and similarly for $R\Pi^*$.

Now, if $A \in \Psi_\rho^{-(d+1)/2-\epsilon}$, it writes as $A = \mathrm{Op}^1(\sigma) + \mathcal{O}(h^\infty \Psi^{-\infty})$, and

$$\chi A \chi \Pi^* = \mathrm{Op}^1(\chi\sigma)\chi\Pi^* + \chi\mathcal{O}(h^\infty\Psi^{-\infty})\chi\Pi^*$$

so that $\chi A \chi \Pi^*$ is HS. □

Proposition 2.1.26. *Take any $\epsilon > 0$. Let $A \in \Psi_\rho^{-d-1-\epsilon}$ with principal symbol a, and χ still supported in $\{y > a\}$. Then $\chi A \chi \Pi^*$ is trace class, and if $d = 1$,*

$$Tr\left[\chi A \chi \Pi^*\right] = \frac{1}{h^{d+1}}\left[\int_{T^*Z} \chi^2(x) a(x, h\xi) + \mathcal{O}(h^{1-2\rho} + h|\log h|)\right].$$

If $d \geqslant 2$, the same holds, but the remainder is $\mathcal{O}(h^{1-2\rho})$.

Observe that in the case dimension 2 ($d = 1$), the remainder is not as good as for compact surfaces.

Proof. First, if $R \in \mathcal{O}(h^\infty)\Psi^{-\infty}$ is supported in $\{y > a\}$, then $R\Pi^*$ and $\Pi^* R$ are trace class since, for example, we can write

$$\Pi^* R = \Pi^* \chi (P+1)^{-N} \Pi^* (P+1)^N R$$

and this is the product of two HS operators.

Now, if $A \in \Psi_\rho^{-d-1-\epsilon}$, we can write

$$A = \mathrm{Op}^1(\tilde{a}) + \mathcal{O}(h^\infty \Psi^{-\infty})$$

with $\tilde{a} = a + \mathcal{O}(h^{1-2\rho} S_\rho^{-d-2-\epsilon})$. Observe ($\sqrt{\chi}$ also is C^∞)

$$\chi\,\mathrm{Op}^1(\tilde{a})\chi\Pi^* = \left[\chi\,\mathrm{Op}^1(\tilde{a})\sqrt{\chi}(P+1)^{(d+1)/2+\epsilon/2}\Pi^*\right]\left[(P+1)^{-(d+1)/2-\epsilon/2}\Pi^*\sqrt{\chi}\right].$$

In the RHS, we have shown in the proof of lemma 2.1.25 that the second term of the product is HS. Using proposition 2.1.23, we can write the first term as $\mathrm{Op}^1(b) + R$ where $b \in S_\rho^{-(d+1)/2-\epsilon/2}$ and R is asymptotically negligible. It is thus also HS, and the product is trace class.

Writing the trace as the integral of the kernel along the diagonal, we obtain

$$Tr\chi A\chi\Pi^* = \frac{1}{(2\pi)^{d+1}} \sum_{J \in \Lambda'^*} \int_{Z\times\mathbb{R}} \chi^2 \tilde{a}(x, h\xi) dx dY + \mathcal{O}(h^\infty).$$

To end the proof, we use:

Lemma 2.1.27. *If $a \in S_\rho^{-d-1-\epsilon}$ is supported in some $\{y > a > 0\}$ and $d \neq 2$,*

$$\sum_{J\in\Lambda'^*} \int_{Z\times\mathbb{R}} a(x, h\xi) dx dY = \frac{1}{h^{d+1}} \left[\int_{T^*Z} a + \mathcal{O}(h^{d-2\rho} + h^d |\log h|) \right].$$

and both sums converge absolutely. When $d = 2$, the remainder is $\mathcal{O}(h^{d-2\rho}|\log h|)$.

□

The proof of lemma 2.1.27:

Proof. Let D' be fundamental domain for the action of Λ' on $\mathbb{R}^d$. Assume D' to be symmetric around 0, and of bounded diameter. Its volume is 1. Then, for $\varpi \in \Lambda'^*$ and $f \in C^2(\mathbb{R}^d)$, using Taylor's formula

$$\left| f(h\varpi) - \frac{1}{h^d} \int_{h\varpi + hD'} f(J) dJ \right| \leqslant C \frac{h^2}{h^d} \int_{h\varpi+hD'} \| d_J^2 f(J) \|.$$

hence, cutting $\mathbb{R}^d$ in $hD' \cup_{\varpi\in\Lambda'^*} (hD' + h\varpi)$, the difference between the two main terms in lemma 2.1.27 is bounded up to some constant by

$$h^{-d-1} \int_{(x,\xi)\in T^*Z, J\in hD'} |a(x,\xi)| + h^{1-d} \int_{(x,\xi)\in T^*Z, J\notin hD'} \| d_J^2 a \|.$$

For the first term, integrate in variable θ (losing a constant $\mathrm{vol}(D) = 1$) and then in variable Y after rescaling. This leads to a bound by

$$h^{-d-1} \int_{J\in hD', y>a} \frac{1}{y} (1 + y^2 |J|^2)^{\frac{-d-\epsilon}{2}}$$

Rescaling the y variable, this is bounded by (note the use of polar coordinates $r = |J|$, and $u = yr$),

$$h^{-d-1} \int_0^{Ch} r^{d-1} dr \int_{ar}^{+\infty} \frac{du}{u(1+u^2)^{\frac{d+\epsilon}{2}}}.$$

This is $\mathcal{O}(h^{-1}|\log h|)$. Likewise for the second term, it is bounded by :

$$h^{1-d-2\rho} \int_{Ch}^{\infty} r^{d-1} dr \frac{1}{r^2} \int_{ar}^{+\infty} \frac{udu}{(1+u^2)^{\frac{d+\epsilon}{2}+1}}$$

This is $\mathcal{O}(h^{-1-2\rho})$ if $d \neq 2$ and $\mathcal{O}(h^{-1-2\rho}|\log h|)$ when $d = 2$. □

2.2. Applications

Now we will present some applications of the cusp-quantization.

2.2.1. Cusp manifolds

2.2.1.1. Quantization

As we said in the introduction, cusp manifolds are described as a compact manifold with boundary to which is glued a finite number of cusps. Here, we give a formal definition that will simplify the construction of the quantization:

Definition 2.2.1. *Let (M, g) be a complete $(d+1)$-dimensional riemannian manifold. M is said to be a cusp manifold if it is endowed with a cusp atlas $\mathcal{F}$, that is*

- *a finite collection $(U_i, U_i', \gamma_i)_i$ of $\mathbb{R}^{d+1}$-charts, that is, diffeomorphisms $\gamma_i : U_i \subset M \to U_i' \subset \mathbb{R}^n$, with U_i relatively compact.*
- *a finite collection $(Z_j, Z_j', \gamma_j^c)_j$ of cusp-charts, that is, diffeomorphisms $\gamma_j^c : Z_j \subset M \to Z_j' \subset Z_{\Lambda_j}$ such that γ_j^c is an isometry, and Z_j' is of the form $\{y > a_j\}$.*

We require that

- *No two Z_j's intersect.*
- *The coordinate changes between two (U_i')'s or Z_j' and U_i' are diffeomorphisms.*
- *The lattices Λ_i have covolume 1. This is a convention, and there is only one choice of height function y_i that is coherent with that choice.*

Definition 2.2.2. *In $\mathbb{R}^{d+1}$, we define the* Kohn-Nirenberg *symbols of order n, in the usual way, as in [Zwo12, p.207]: $\sigma \in S^n_\rho(\mathbb{R}^{d+1})$ whenever for all $k, k' \geqslant 0$ there is a constant $C_{k,k'} > 0$,*

$$|d_x^{k'} d_\xi^k \sigma| \leqslant C_{k,k'} h^{-\rho(k+k')} \langle \xi \rangle^{n-k}, \text{ for } x, \xi \in \mathbb{R}^{d+1}$$

The class $S^n_\rho(M)$ of hyperbolic symbols of order (n, ρ) is composed of the functions σ on T^*M such that for any chart $(U, V \subset N, \gamma)$ in the atlas, the function

$$\sigma^U(x, \xi) := \gamma^* \sigma \left[= \sigma(\gamma^{-1}(x), d\gamma(\gamma^{-1}(x))^* \xi)\right]$$

is the restriction to T^*V of some element of $S^n_\rho(N)$ (with $N = \mathbb{R}^{d+1}$ or $N = Z_{\Lambda_i}$). The invariance by coordinate changes of the Kohn-Nirenberg class [Zwo12, theorem 9.4, p.207] implies that this is well defined — it does not depend on the choice of the atlas. We also let $S^{-\infty}_\rho = \cap_{n \in \mathbb{Z}} S^n_\rho$.

To define a quantization on cusp manifolds that enjoys all usual properties, we follow the procedure in p. 347 through to p. 352 in [Zwo12]. A pseudo-differential operator on M is defined as an operator $C_c^\infty(M) \to C^\infty(M)$ such that restricted to any chart, it is pseudo-differential — in the case of a cusp-chart, this means that it is in some $\Psi_\rho(Z_{\Lambda_i})$. We also require that they are pseudo-local — that is, when we truncate their kernel at a fixed distance of the diagonal, we obtain negligible operators.

Lemma 2.1.8 proves that it suffices to check the above properties for the finite set of charts of some cusp-atlas. Lemmas 2.1.8 and 2.1.15 ensure that the class of pseudors is not reduced to compactly supported operators, because pull backs of elements of $\Psi(Z_{\Lambda_i})$ *are* pseudo-local.

We can define the semi-classical principal symbol $\sigma^0(A)$ of a pseudor A as for pseudors on compact manifolds — once again thanks to lemma 2.1.8 — and according to the definition, $\sigma^0(A)$ is in some $S^n_\rho(M)/h^{1-2\rho}S^{n-1}_\rho$. The class of A such that $\sigma^0(A) \in S^n_\rho(M)/h^{1-2\rho}S^{n-1}_\rho$ is denoted Ψ^n_ρ. We let $\Psi^{-\infty}_\rho(M)$ be the class of smoothing operators in the Sobolev sense — as in definition 2.1.13. Let $\Psi_\rho(M) = \cup_{n\geqslant -\infty}\Psi^n_\rho$. When we omit the ρ, we refer to the case $\rho = 0$.

Using charts, our quantization Op in Z_{Λ_i} and the usual Weyl quantization on $\mathbb{R}^{d+1}$, we are able to build a quantization procedure Op on M, that is, a section of the symbol map. Using classical results, and the first part of the article, we see that Ψ^0_ρ gives bounded operators on $L^2(M)$, whose norm is the L^∞ norm of the symbol, up to a $o(1)$ term as $h \to 0$ — that could be estimated with derivatives of the symbol.

For a height a bigger than all the a_j, we define Π^*_a as the projection on non zero Fourier modes in $\{y > a\}$ (truncating in *all* cusps):

$$\Pi^*_a f := f - \mathbb{1}(y > a)\int f d\theta.$$

The following hold :

Proposition 2.2.3. *Let $A \in \Psi^{-1}_\rho(M)$. Then $\Pi^*_a A$ is compact on L^2.*
Let $A \in \Psi^{-n}_\rho(M)$ with $2n > d + 1$. Then $\Pi^*_a A \Pi^*_a$ is Hilbert-Schmidt.
Let $A \in \Psi^{-n}_\rho(M)$ with $n > d + 1$. Then $\Pi^*_a A \Pi^*_a$ is trace class, and

$$Tr \Pi^*_a A \Pi^*_a = \frac{1}{h^{d+1}}\left[\int_{T^*M} \sigma(A) + \mathcal{O}(h^{1-2\rho} + h|\log h|)\right].$$

This remainder is valid when $d \neq 2$. When $d \geqslant 2$, it is $\mathcal{O}(h^{1-2\rho})$.

Proof. In the compact part of M, these are classical results — see theorem 4.28 p. 89, and remark (C.3.6) p. 412 in [Zwo12]; see also proposition 9.2 and theorem 9.5 p.112 and following in [DS99]. So we only need to prove this when A is only supported in the cusps, and for negligible operators. For the Hilbert-Schmidt and the trace-class property, this are the contents of lemma 2.1.25 and proposition 2.1.26 when A is supported only in the cusps.

As to negligible operators, they can always be written down as the product of another negligible operator with some power of $(P+1)^{-1}$, and the arguments we used in the proof of lemma 2.1.25 and proposition 2.1.26 will carry on, so that what we really need to prove is that $\Pi^*_a A$ is compact on $L^2(M)$ when $A \in \Psi^{-1}_\rho$.

But that is a consequence of the fact that $\Pi^*_a H^1(M)$ is compactly injected in $L^2(M)$. Once again, as this is always true for compact manifolds with boundary (Rellich's theorem), it suffices to prove it for for the cusps. More precisely, we need to show that $\{\mathbb{1}_{y>a} f \mid f \in H^1(Z_\Lambda), \Pi^* f = f\}$ is compactly injected in $L^2(Z_\Lambda)$. We recall the proof from [LP76] — see pp. 206 and following. Consider the fact that $f \in H^1(Z_\Lambda) \mapsto \mathbb{1}_{a<y<T}\Pi^* f \in L^2(Z_\Lambda)$ is compact. Now, using the Wirtinger inequality in the torus, one can prove that if $\Pi^* f = f$,

$$\|\mathbb{1}_{y>T} f\|_{L^2(Z_\Lambda)} \leqslant \frac{C}{T^{d+1}}\|\mathbb{1}_{y>T}\nabla f\|_{L^2(Z_\Lambda)}.$$

This proves that the mapping $\mathbb{1}_{y>a}\Pi^* : H^1(Z_\Lambda) \to L^2(Z_\Lambda)$ is the norm limit of a sequence of compact operators, so it is compact. □

2.2.1.2. Egorov lemma for Ehrenfest times

In this section, we give an Egorov lemma up to Ehrenfest time, which was the original motivation for what we have done so far. We need to explain how we measure distances on T^*M. The vertical subbundle V in TT^*M is the kernel of $T\pi$ where $\pi : T^*M \to M$ is the projection on the base. The riemannian metric on M gives a horizontal bundle H in TT^*M that is transverse to V. There are natural identifications between respectively V and T^*M, H and TM. The only metric on T^*M that renders V orthogonal to H and that makes those identifications isometries is called the *Sasaki metric*. It is in some sense the natural metric to use on T^*M for our problem; we recall a few facts on it in appendix A.3. Now that we have specified a riemannian metric on T^*M, we can define the spaces $\mathscr{C}^k(T^*M)$ as in appendix A.3. Let us introduce a particular class of symbols. Recall that $p = |\xi|_x^2/2$:

Definition 2.2.4. *Let U be some open set of $\mathbb{R}^2$. For $E > 0$, let S_C^E denote the class of functions σ on $U \times T^*M$ that are $\mathscr{C}^\infty(T^*M)$ in the second variable, supported in $(T^*M)_E := \{p \leqslant E\}$. Additionally require there are constants $C_k > 0$ such that*

$$\|\sigma(h, \tau; .)\|_{\mathscr{C}^k(T^*M)} \leqslant C_k e^{Ck|\tau|}$$

where (h, τ) are the coordinates in $\mathbb{R}^2$. This defines a topology on S_C^E.

Remark 2.4. *From proposition A.3.2, elements of S_C^E are symbols in $S^{-\infty}$ for fixed τ. Additionally, if the open set U is $\{C|\tau| \leqslant \rho |\log h|\}$ with $\rho < 1/2$, elements of S_C^E are symbols in $S_\rho^{-\infty}$, and can be quantized. We will assume that U takes this form in the rest of the article.*

Let us point out that when A is in some $\Psi^n(M)$, and $\sigma \in S_C^E$, up to a negligible operator R,

$$[A, \mathrm{Op}(\sigma)] - \frac{h}{i} \mathrm{Op}(\{\sigma(A), \sigma\}) = \mathrm{Op}(\tilde{\sigma}) + R$$

where $\tilde{\sigma}$ is $\mathcal{O}((he^{C|\tau|})^2)$ in S_C^E.

Let us introduce

Definition 2.2.5. *The* maximal Lyapunov exponent *of the geodesic flow on $(T^*M)_E$ is defined as*

$$\lambda_{max}(E) := \sup_{\xi \in (T^*M)_E} \limsup_{t \to \infty} \frac{1}{|t|} \log \|d_\xi \varphi_t\|.$$

The norm is measured with the Sasaki metric.

Using Jacobi fields and Rauch's comparison theorem — see 1.28 in section 1.10 of [CE08] — one can prove that $\lambda_{max}(E)$ is bounded by $E\kappa$ where $-\kappa$ is the minimum of the curvature of M. Observe that proposition A.4.1 implies that for any $\lambda > \lambda_{max}(E)$, and any $f \in \mathscr{C}^\infty(T^*M)$ supported in $(T^*M)_E$, $\{f \circ \varphi_t\}$ is in S_λ^E.

Recall that the Schrödinger propagator is

$$U(t) = e^{-itP/h} = e^{ith\Delta/2}.$$

We have

Theorem 2.3. *Let $\sigma \in \mathscr{C}^\infty(T^*M)$ be supported in $(T^*M)_E$. Then, for any $\rho < 1/2$ and any $\lambda > \lambda_{max}(E)$, there exists a symbol $\tilde{\sigma}_\rho$ that is in S^E_λ, with $U = \{|\tau| \leqslant \rho|\log h|/\lambda\}$. On U,*

$$\tilde{\sigma}(t,x,\xi) = \sigma(\varphi_t(x,\xi)) + \mathcal{O}(h|t|e^{2\lambda|t|}),$$

and

$$U(-t)\operatorname{Op}(\sigma)U(t) = \operatorname{Op}(\tilde{\sigma}) + \mathcal{O}((|t|he^{2\lambda|t|})^\infty)$$

where the remainder is asymptotically smoothing.

Since Beal's theorem — see theorem 8.3 in [Zwo12], it is a criterion to prove that an operator is *actually* pseudo-differential — is not available to us, we can only prove that the remainder is *asymptotically* smoothing.

Proof. Let us assume that we found an exact solution $\tilde{\sigma}$. Then, we would have :

$$\operatorname{Op}(\tilde{\sigma}) = e^{itP/h}\operatorname{Op}(\sigma)e^{-itP/h}$$

i.e

$$\operatorname{Op}(\sigma) = e^{-itP/h}\operatorname{Op}(\tilde{\sigma})e^{itP/h}.$$

Differentiating with t,

$$0 = e^{-itP/h}\left[\operatorname{Op}(\partial_t\tilde{\sigma}) - \frac{i}{h}[P,\operatorname{Op}(\tilde{\sigma})]\right]e^{itP/h}.$$

This is the equation that we try to solve. All along our development, we will follow the proof in [Zwo12] closely. Let us build by induction a family of operators

$$B_n(t) = \operatorname{Op}(b_n) \quad , \quad E_n(t) = \operatorname{Op}(e_n)$$

where b_n and e_n are in S^E_λ, satisfying :

$$\begin{aligned} \frac{h}{i}\partial_t B_n &= [P,B_n] + E_n + R_n. \\ B_n(0) &= \operatorname{Op}(\sigma) \end{aligned}$$

the remainder R_n being negligible, and with the estimates :

$$\begin{aligned} b_n - b_{n-1} &= \mathcal{O}_{S^E_\lambda}((|t|h)^n e^{2n\lambda|t|}) \text{ for } n > 0 \\ e_n &= \mathcal{O}_{S^E_\lambda}(h^{2+n}|t|^n e^{(2n+2)\lambda|t|}). \end{aligned}$$

For $n = 1$, define

$$B_0 = \operatorname{Op}(\sigma\circ\varphi_t).$$

This is $\mathcal{O}(1)$ in S^E_λ. Then

$$\begin{aligned} \frac{h}{i}\partial_t B_0 &= \frac{h}{i}\operatorname{Op}(\{p,\sigma\circ\varphi_t\}) \\ &= [P,B_0] + E_0 + R_0, \end{aligned}$$

where R_0 is negligible and $E_0 = \mathrm{Op}(e_0)$. From the product formula, we get that e_0 is still supported in $(T^*M)_E$, and it is $\mathcal{O}(h^2 e^{2\lambda|t|})$ in S^E_λ.

Proceeding by induction, assume that we have built $b_1, \dots, b_n$, $e_1, \dots, e_n$ and $c_1, \dots, c_n$, for some $n \geqslant 0$, and let

$$c_{n+1} = \frac{i}{h}\int_0^t e_n(s) \circ \varphi_{t-s} ds.$$

We have $c_{n+1} = \mathcal{O}((|t|h)^{n+1} e^{(2n+2)\lambda|t|})$ in S^E_λ. Let $C_{n+1} = \mathrm{Op}(c_{n+1})$. One gets

$$\begin{aligned} \frac{h}{i}\partial_t C_{n+1} &= \frac{h}{i}\mathrm{Op}(\{p, c_{n+1}\} + \frac{i}{h} e_n) \\ &= [P, C_{n+1}] + E_n - E_{n+1} + R_{n+1} \end{aligned}$$

where

$$E_{n+1} = \mathrm{Op}(e_{n+1}) \text{ with } e_{n+1} = O_{S^E_\lambda}(h^{3+n}|t|^{n+1} e^{(2n+4)\lambda|t|})$$

At last, define

$$B_{n+1} = B_n - C_{n+1} = \mathrm{Op}(b_n - c_{n+1}).$$

Such b_{n+1} and e_{n+1} satisfy the announced properties. Now, since

$$\frac{h}{i}\partial_t[e^{-itP/h} B_n e^{itP/h}] = e^{-itP/h}(E_n + R_n) e^{itP/h}$$

Duhamel's formula gives :

$$\begin{aligned} B_n(t) &= e^{itP/h}\,\mathrm{Op}(\sigma) e^{-itP/h} \\ &\quad - \frac{i}{h}\int_0^t U(s-t)[E_n(s) + R_n(s)]U(t-s) ds. \end{aligned}$$

Since the operators $U(t)$ are bounded from H^s to H^s for any s, we deduce that the second term in the RHS is $\mathcal{O}((|t|he^{2\lambda|t|})^{n+1} h^{-2N})$ in $H^{-N} \to H^N$ operator norm, which is the type of estimates we seek on the remainder.

From the above, we deduce that we can find a symbol $\tilde{\sigma} \in S^E_\lambda$ by Borel summation such that as $h \to 0$

$$\tilde{\sigma} \sim b_0 - \sum_1^\infty c_n = \sigma \circ \varphi_t + \mathcal{O}_{S^E_\lambda}(h|t|e^{2\lambda|t|}),$$

and such that $\tilde{\sigma}$ satisfies the condition of the theorem.

□

Remark 2.5. *Following the support of the b_n's, e_n's, we find that $\tilde{\sigma}$ is exactly supported in $\varphi_t(\mathrm{supp}(\sigma))$. Actually, the whole operator is microsupported on that set ; if we multiply our conjugated operator by some $\mathrm{Op}(\eta)$ such that η vanishes on $\mathrm{supp}\,\tilde{\sigma}$, we obtain a negligible operator (not only asymptotically).*

2.2.2. Extending a result of Semyon Dyatlov

2.2.2.1. Spectral theory and Eisenstein functions

The following facts on the spectral theory of the Laplacian on cusp-manifolds are contained in [Mül83]. However, in that article, Müller considered cusps where the horizontal slices were arbitrary compact d-dimensional manifolds instead of tori, so that his definition of *Riemannian manifolds with cusps* is more general than our cusp-manifolds. However, he also wrote an article in the case of surfaces [Mül92] with the same definition of cusp, which is a good place to start if one wants to learn about the spectral theory of the Laplacian of cusp surfaces.

The non-negative Laplacian $-\Delta$ acting on $C_c^\infty(M)$ functions has a unique self-adjoint extension to $L^2(M)$ and its spectrum consists of

1. Absolutely continuous spectrum $\sigma_{ac} = [d^2/4, +\infty)$ with multiplicity κ (the number of cusps).
2. Discrete spectrum $\sigma_d = \{\lambda_0 = 0 < \lambda_1 \leqslant \cdots \leqslant \lambda_i \leqslant \ldots\}$, possibly finite, and which may contain eigenvalues embedded in the continuous spectrum. To $\lambda \in \sigma_d$, we associate a family of orthogonal eigenfunctions that generate its eigenspace $(u_\lambda^i)_{i=1\ldots d_\lambda} \in L^2(M) \cap C^\infty(M)$.

The generalized eigenfunctions associated to the absolutely continuous spectrum are the Eisenstein functions, $E(x,s) = (E_i(x,s))_{i=1\ldots k}$. Each E_i is a meromorphic family (in s) of smooth functions on M. Its poles are contained in the open half-plane $\{\Re s < d/2\}$ or in $(d/2, 1]$ and are called *resonances*. The Eisenstein functions are characterized by two properties :

1. $\Delta_g E_i(.,s) = s(d-s)E_i(.,s)$
2. In the cusp Z_j, $j = 1\ldots k$, the zeroth Fourier coefficient of E_i in the θ variable equals $\delta_{ij} y_j^s + \phi_{ij}(s) y_j^{d-s}$ where y_j denotes the y coordinate in the cusp Z_j and $\phi_{ij}(s)$ is a meromorphic function of s.

Definition-Proposition 2.2.6. *A* resonant state *on M for the spectral parameter s is a solution u on M of $-\Delta u = s(d-s)u$ that satisfies the following conditions. Its zeroth Fourier coefficient in each cusp is proportional to y^s — there is no y^{d-s} term. The non-zero Fourier modes are L^2.*

All such functions can be obtained in the following way. Consider $\phi(s)$ the scattering matrix *obtained by gathering the coefficients $\phi_{ij}(s)$. The symmetries of the situation imply that $\phi(s)\phi(d-s) = \mathbb{1}$. One can show (see [Mül83, Mül92]) that the poles of ϕ are exactly the poles of E.*

If there is a resonant state for the parameter s, s has to be a resonance. Then $\phi(1-s)$ is not invertible; take $\mathbf{u} \neq 0$ a vector in its cokernel. The function on M given by

$$u = \mathbf{u}_1 E_1(1-s) + \cdots + \mathbf{u}_\kappa E_\kappa(1-s) \tag{2.21}$$

is a resonant state for the resonance s. Reciprocally, one can check that they are all obtained in this way (by the uniqueness property of the Eisenstein series).

The set of resonant states for a resonance s replace the eigenspace for a L^2 eigenvalue. However, since no resonant state can be L^2, there is no obvious way to normalize them.

Let us recall the construction of the Eisenstein functions. First, we can always reduce the size of the cusps, so that they take the form $\{y_i \geqslant a\}$ with $a \geqslant 1$ not depending on i. On M we define a function y_M that corresponds to y_i on $Z_i \cap \{y_i \geqslant 2a\}$, and equals 1 on M_0. Let χ be a smooth non-decreasing function that equals 1 on $[3a, +\infty[$, and vanishes on $]-\infty, 2a]$. We let χ_i be the function supported in cusp Z_i, where it is $\chi \circ y_i$. Take $E^0(s,x) = y_M^s$. Then consider

$$E_i(s,x) := \chi_i E^0(s,x) + (-\Delta - s(d-s))^{-1}[\Delta, \chi_i]E^0(s,x).$$

Since χ' is compactly supported, $[\Delta, \chi_i]E^0(s,.)$ is compactly supported and in L^2, so this is well defined for $\Re s > d/2$, and of the announced form. One can check that

$$(-\Delta - s(d-s))E_i = -[\Delta, \chi_i]E^0 + [\Delta, \chi_i]E^0 = 0.$$

to see that the E_i's satisfy the announced properties. Uniqueness is then straightforward.

In what follows, we will use the notation:

$$s = d/2 \pm i/h + \eta(h) \text{ with } \eta > 0. \tag{2.22}$$

Let us define the measures $\mu_{i,\eta}$ announced in the introduction. For $f \in C^0_c(T^*M)$ compactly supported, let

$$\mu^{\pm}_{i,\eta}(f) := 2\eta a^{2\eta} \int_{\mathbb{R}\times\mathbb{T}_\Lambda} e^{-2\eta t} f \circ \varphi_{\mp t}(a, \theta, \pm 1/a, 0) dt d\theta. \tag{2.23}$$

This defines two Radon measures. We also recall the definition of the Wigner distributions

$$\langle \mu^h_{i,j}(s), \sigma\rangle := \langle \mathrm{Op}(\sigma)E_i(s), E_j(s)\rangle \text{ for } \sigma \in C^\infty_c(T^*M).$$

We will prove the following theorem

Theorem 2.4. *Consider $s_h = d/2 \pm i/h + \eta(h)$. All the limits are taken when $h \to 0$.*

1. *If $\eta(h) \to \nu > 0$, then $\eta\mu^h_{i,j}(s_h) \rightharpoonup \delta_{i,j}\pi\mu^{\pm}_{i,\nu}$ in $C^\infty_c(T^*M)'$.*
2. *Assume that M has negative curvature, and $\eta \to 0$ with*

$$\liminf \ \eta \frac{|\log h|}{\log|\log h|} > \lambda_{max}\left(\frac{1}{2}\right),$$

where λ_{max} is the lyapunov exponent defined in definition 2.2.5. Then $\eta\mu^h_{i,j}(s_h) \rightharpoonup \delta_{i,j}\pi\mathscr{L}_1$.

The case when $\eta \to \nu > 0$ was proven in dimension 2 by Semyon Dyatlov in [Dya12]. We cautiously follow the steps of his proof, paying attention to the constants. The long time Egorov lemma is really what enables us to extend S. Dyatlov's result and get (2).

The proof is divided into three parts. We first approximate the Eisenstein series by a Lagrangian state propagated by the Schrödinger flow. Such an approximation cannot work near the spectrum, and that is why the approach taken here probably cannot be improved to capture resonances arbitrarily close to the spectrum. Then, we use the Egorov lemma to reduce the problem to a stationary phase computation in the cusp. The last part of the proof is a dynamical argument, from Babillot; we essentially prove that incoming horocycles from the cusp equidistribute in M.

It suffices to consider the case $\Im s \to +\infty$, the other can be deduced thereof. In the following, η will be bounded, $1/\eta$ a priori tempered with respect to h and σ a given $C^\infty_c(T^*M)$ function.

2.2.2.2. Reduction to a lagrangian expression

In this section, we will use the Egorov lemma several times. First

Remark 2.6. *We fix an exponent* $\lambda > \lambda_{max}(1/2)$*. Observe that since the hamiltonian* p *of the geodesic flow is 2-homogeneous,* $\varphi_t(k\xi) = k\varphi_{kt}(\xi)$*. Consider* $\Phi_k : T^*M \to T^*M$ *the fiber-wise multiplication by* k*. Then*

$$d\varphi_t = d\Phi_k \circ d\varphi_{kt} \circ d\Phi_k^{-1}.$$

If $k \geqslant 1$*, we have* $\|d\Phi_k\| = k$ *and* $\|d\Phi_k^{-1}\| = 1$ *(by inspecting the behaviour of* $d\Phi_k$ *on the vertical and horizontal bundles of* T^*M*). We deduce that*

$$\lambda_{max}(kE) = k\lambda_{max}(E).$$

It entails that for any $\epsilon > 0$ *sufficiently small,* $\lambda > \lambda_{max}(E = 1/2 + \epsilon)$*. In the following, we fix such an* $\epsilon > 0$*.*

Let us take $T > 0$ such that σ is supported in $\{y_M \leqslant ae^T\}$. We aim to replace $E_i(s)$ on the support of σ by a propagated incoming wave. That is why we define the following. Let $\tilde{\chi} \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\tilde{\chi} \equiv 1$ on $]-\infty, \ln 4]$ and $\tilde{\chi} \equiv 0$ on $[\ln 5, +\infty[$. For $\tau \in \mathbb{R}^+$, let

$$\chi_\tau := \tilde{\chi}\left(\ln\left(\frac{y_M}{a}\right) - \tau\right). \tag{2.24}$$

This cutoff function varies for $4ae^\tau \leqslant y_M \leqslant 5ae^\tau$. Also write:

$$W = \frac{h^2}{2}s(d-s) = \frac{h^2}{2}\left[\frac{d^2}{4} + \frac{1}{h^2} - \eta^2 - 2i\frac{\eta}{h}\right] \tag{2.25}$$

$$= \frac{1}{2}\left[1 - 2i\eta h + h^2\left(\frac{d^2}{4} - \eta^2\right)\right]. \tag{2.26}$$

Recall $E^0 = y_M^s$. In this paragraph, we prove that

Lemma 2.2.7. *Let* $t = t_0|\log h|/(2\lambda)$*, where* $0 < t_0 < 1$*, and assume*

$$\eta \geqslant C_\lambda \frac{\log|\log h|}{|\log h|}$$

with $C_\lambda > \lambda/t_0 > \lambda_{max}(1/2)$*. Then*

$$\eta\langle \mathrm{Op}(\sigma)E_i, E_i\rangle = \eta e^{-2\eta t}\langle \mathrm{Op}(\sigma_{\epsilon,t})\chi_{T+t}\chi_i E^0, \chi_{T+t}\chi_i E^0\rangle + o_{h\to 0}(1). \tag{2.27}$$

In the equation above, $\sigma_{\epsilon,t} = \sigma_\epsilon \circ \varphi_{-t}$ *where* σ_ϵ *is a symbol supported in* $\{|p - 1/2| < \epsilon\}$*, and coincides with* σ *in a neighbourhood of* $\{p = 1/2\}$*. For* $i \neq j$*, we also have*

$$\eta\langle \mathrm{Op}(\sigma)E_i, E_j\rangle = o_{h\to 0}(1). \tag{2.28}$$

Proof. For now, we assume only $t \geqslant 0$. Let us introduce some notations.

$$\tilde{E}_i^0(s,t) := \chi_{T-\ln 3}e^{\frac{i}{h}t(P-W)}\chi_{T+t}\chi_i E^0(s),$$
$$\tilde{E}_i(s,t) := \chi_{T-\ln 3}e^{\frac{i}{h}t(P-W)}\chi_{T+t}E_i(s).$$

and prove :

Lemma 2.2.8. *When η remains bounded,*

$$\|\chi_{T-\ln 3}E_i(s) - \tilde{E}_i^0(s,t)\|_{L^2} = \mathcal{O}\left(\frac{e^{-\eta t}}{\eta}\right) + \mathcal{O}((|t|he^{2\lambda|t|})^\infty).$$

Proof. We write

$$\chi_{T-\ln 3}E_i - \tilde{E}_i^0 = (\chi_{T-\ln 3}E_i - \tilde{E}_i) + (\tilde{E}_i - \tilde{E}_i^0).$$

Then, we prove successively

Lemma 2.2.9.

$$\tilde{E}_i - \tilde{E}_i^0 = \mathcal{O}_{L^2}\left(\frac{e^{-\eta t}}{\eta}\right).$$

and

Lemma 2.2.10.

$$\chi_{T-\ln 3}E_i - \tilde{E}_i = \mathcal{O}_{L^2}((|t|he^{2\lambda|t|})^\infty).$$

□

we start with lemma 2.2.9.

Proof. We have

$$\tilde{E}_i - \tilde{E}_i^0 = \chi_{T-\ln 3}e^{\frac{it}{h}(P-W)}\chi_{T+t}(-\Delta - s(d-s))^{-1}[\Delta, \chi_i]E^0$$

Thus

$$\|\tilde{E}_i - \tilde{E}_i^0\|_{L^2} \leqslant e^{-\Re\frac{itW}{h}}\|(-\Delta - s(d-s))^{-1}\|_{L^2\to L^2}\|[\Delta, \chi_i]E^0\|_{L^2}.$$

since Δ is self adjoint, we have the obvious resolvent estimate

$$\|(-\Delta - s(d-s))^{-1}\|_{L^2\to L^2} \leqslant \frac{h}{2\eta}.$$

What is more, $\Re(itW) = h\eta t$. Now,

$$[\Delta, \chi_i]E^0 = (\Delta\chi_i)E^0 + 2ys\partial_y\chi_i E^0 = \mathcal{O}_{L^2}\left(\frac{1}{h}\right).$$

putting all three inequalities together, we conclude. □

we go on to lemma 2.2.10.

Proof. When $t = 0$,

$$\tilde{E}_i(0) = \chi_{T-\ln 3}E_i$$

because $\chi_{T-\ln 3}\chi_T = \chi_{T-\ln 3}$ $(s + \ln 3 \leqslant \ln 5 \Rightarrow s \leqslant \ln 4)$. For $\tau \in [0,t]$, let

$$A(\tau) = \chi_{T-\ln 3}e^{\frac{i\tau}{h}(P-W)}\chi_{T+t}E_i$$
$$\frac{d}{d\tau}A = \chi_{T-\ln 3}e^{\frac{i\tau}{h}(P-W)}\frac{i}{h}[P, \chi_{T+t}]E_i.$$

We want to use Egorov's lemma, first we need to localize the expression. Start with localization in the energy variable. Let $f \in C_c^\infty(\mathbb{R})$ be so that f is supported at distance less than ϵ of $1/2$ and equals 1 near $1/2$. Let

$$F = \mathrm{Op}(f \circ p).$$

F is a parametrix for $f(P)$, but we do not directly use that fact. We claim that

$$(1 - F)[P, \chi_{T+t}]E_i(s, .) = \mathcal{O}_{L^2}(h^\infty) \tag{2.29}$$

First, remark that $f \circ p$ is indeed a symbol in the class $S_0^{-\infty}$. By ellipticity, we can solve

$$1 - F = \mathrm{Op}(r_n)(P - 1/2)^n + \mathcal{O}_{L^2 \to L^2}(h^\infty)$$

for all $n \in \mathbb{N}$, with symbols r_n in S_0^{-2n}. Observe that

$$(P - 1/2)E_i = (W - 1/2)E_i = \mathcal{O}(\eta h)E_i.$$

and

$$(P - 1/2)^n\,[P, \chi] = \sum_{k=0}^{n} \binom{n}{k} P^{[k+1]}[\chi]\,(P - 1/2)^{n-k}$$

where $P^{[k]}[\chi] = [P, [P, \ldots, [P, \chi] \ldots]$ with k occurences of P. From the proof of lemma 2.2.9, we know that the L^2 norm of $E(s, .)$ restricted to any compact set is $\mathcal{O}(1/\eta)$. Now, since $r_n \in S_0^{-2n}$, $\mathrm{Op}(r_n)P^k[\chi]$ (for $k \leqslant n + 1$) is bounded on L^2 with norm h^k, and is compactly supported; the claim (2.29) follows since η is bounded.

Now, we have localized our formulae in the momentum variable :

$$\frac{d}{d\tau}A = \chi_{T-\ln 3}e^{\frac{i\tau}{h}(P-W)}\left(\frac{i}{h}F[P, \chi_{T+t}]E_i + O_{L^2}(h^\infty)\right). \tag{2.30}$$

According to the support hypothesis we have made, we can pick a function $g \in C_c^\infty(\mathbb{R})$ such $g \circ y_M \equiv 1$ on a neighbourhood of $\mathrm{supp}(\partial_y \chi_{T+t})$, and that for all $0 < \tau < t$, $\varphi_{-\tau}(\mathrm{supp}(f \circ p \times g \circ y_M))$ does not intersect the δ-neighbourhood of $\mathrm{supp}(\chi_{T-\ln 3})$ where δ is some positive number.

We can insert $1 = g + 1 - g$ in (2.30) between F and $[P, \chi_{T+t}]$. Now, theorem 2.3 and remark 2.5 give that,

$$\chi_{T-\ln 3}e^{\frac{i\tau}{h}(P-W)}Fg = e^{-\eta\tau}O_{L^2 \to L^2}((|t|he^{2\lambda|t|})^\infty)$$

Since $\|[P, \chi_{T+t}]E_i\|_{L^2}$ is bounded by some finite power of h, we can conclude the proof of lemma 2.2.8. □

We deduce the following lemma :

Lemma 2.2.11. *For $\epsilon > 0$ small enough, there is a symbol σ_ϵ that is supported at distance $\leqslant \epsilon$ of the energy shell $\{p = 1/2\}$, and coincides with σ on the neighbourhood $\{1/2 - \epsilon/2 \leqslant p \leqslant 1/2 + \epsilon/2\}$, such that for $t > 0$*

$$\langle \mathrm{Op}(\sigma)E_i, E_j\rangle = \langle \mathrm{Op}(\sigma_\epsilon)\tilde{E}_i^0, \tilde{E}_j^0\rangle + \mathcal{O}\left(\frac{e^{-\eta t}}{\eta^2}\right) + \mathcal{O}((|t|he^{2\lambda|t|})^\infty) + \mathcal{O}(h^\infty).$$

Proof. We claim that the quantity in the LHS of the equation is well defined. We only have to prove that $\mathrm{Op}^{\delta}(\sigma) := y_M^{\delta}\,\mathrm{Op}(\sigma) y_M^{\delta}$ is bounded on L^2 for some $\delta > 0$ since then $y^{-\delta} E_i \in L^2(M)$. It suffices to prove it in the cusps. A simple computation shows that in Z_Λ, using the notation 2.1, with $\varsigma \in S^n_\rho(Z_\Lambda)$, $y^\delta \,\mathrm{Op}(\varsigma) y^\delta = \mathrm{Op}(y^{2\delta}\varsigma)_\zeta$ with $\zeta(x) = 4^\delta(x^2+4)^{-\delta}$. Since σ is compactly supported, $y_M^{2\delta}\sigma$ still is a symbol, and $\mathrm{Op}^\delta(\sigma)$ is bounded on L^2.

From the pseudo-locality properties of $\mathrm{Op}(\sigma)$, and the bound $\|y^{-\epsilon}E_i\|_{L^2} = \mathcal{O}(1/\eta)$, we know that

$$\langle \mathrm{Op}(\sigma) E_i, E_j\rangle = \langle \mathrm{Op}(\sigma)\chi_{T-\ln 3}E_i, \chi_{T-\ln 3}E_j\rangle + \mathcal{O}(h^\infty). \tag{2.31}$$

We use the same trick as in the previous proof : we introduce $1 = F + (1-F)$ with $F = \mathrm{Op}(f \circ p)$, where f is smooth, supported in $[1/2-\epsilon, 1/2+\epsilon]$, and equals 1 on $[1/2-\epsilon/2, 1/2+\epsilon/2]$. Then for the same reasons as above

$$\mathrm{Op}(\sigma)\chi_{T-\ln 3}E_i = \mathrm{Op}(\sigma) F \chi_{T-\ln 3}E_i + \mathcal{O}_{L^2}(h^\infty).$$

But then $\mathrm{Op}(\sigma)F = \mathrm{Op}(\sigma_\epsilon) + R$ where R is a negligible operator and σ_ϵ is as announced. From there and 2.31:

$$\begin{aligned}\left|\langle \mathrm{Op}(\sigma)E_i, E_j\rangle - \langle \mathrm{Op}(\sigma_\epsilon)\tilde{E}_i^0, \tilde{E}_j^0\rangle\right| \leqslant\ & |\langle \mathrm{Op}(\sigma_\epsilon)(\chi_{T-\ln 3}E_i - \tilde{E}_i^0), \chi_{T-\ln 3}E_j\rangle| \\ & + |\langle \mathrm{Op}(\sigma_\epsilon)\chi_{T-\ln 3}E_i, \chi_{T-\ln 3}E_j - \tilde{E}_j^0\rangle| \\ & + |\langle \mathrm{Op}(\sigma_\epsilon)(\chi_{T-\ln 3}E_i - \tilde{E}_i^0), \chi_{T-\ln 3}E_j - \tilde{E}_j^0\rangle| \\ & + \mathcal{O}(h^\infty)\end{aligned}$$

We can conclude using lemma 2.2.8, and :

$$\|\chi_{T-\ln 3}E_i\|_{L^2} \leqslant C + \|(-\Delta - s(d-s))^{-1}[\Delta, \chi_i]E^0\|_{L^2} \leqslant C(1 + \frac{1}{\eta})$$

(Recall $1/\eta$ is tempered). □

We finish the proof of lemma 2.2.7. We write :

$$\langle \mathrm{Op}(\sigma_\epsilon)\tilde{E}_i^0, \tilde{E}_j^0\rangle = e^{-\frac{itW}{h} + \frac{it\overline{W}}{h}} \langle A\chi_{T+t}\chi_i E^0, \chi_{T+t}\chi_j E^0\rangle$$

where, again with the notation $U(t) = e^{-\frac{itP}{h}}$,

$$A = U(t)\chi_{T-\ln 3}\,\mathrm{Op}(\sigma_\epsilon)\chi_{T-\ln 3}U(-t).$$

Here again, Egorov's lemma gives

$$A = \mathrm{Op}(\sigma_\epsilon \circ \varphi_{-t}) + \mathcal{O}_{L^2\to L^2}(h|t|e^{2\lambda|t|}).$$

Actually, when $i \neq j$, $\chi_{T+t}\chi_i E^0$ and $\chi_{T+t}\chi_j E^0$ have a distinct support, so that remark 2.5 implies that when $i \neq j$,

$$\eta\langle \mathrm{Op}(\sigma)E_i, E_j\rangle = \mathcal{O}\left(\frac{e^{-\eta t}}{\eta}\right) + \mathcal{O}(\eta(|t|he^{2\lambda|t|})^\infty) + \mathcal{O}(\eta h^\infty). \tag{2.32}$$

This proves the lemma for $i \neq j$, by taking the value $t = t_0|\log h|/(2\lambda)$. Now, we assume that $i = j$, unless specifically stated. We denote $\sigma_{\epsilon,t} = \sigma_\epsilon \circ \varphi_{-t}$. We claim that when η

remains bounded and $\eta \times t \to \infty$, there are constants C_1 and C_2 (depending on T) such that

$$\frac{C_2}{\eta} \leqslant e^{-2\eta t}\|\chi_{T+t}\chi_i E^0\|_{L^2}^2 \leqslant \frac{C_2}{\eta}. \tag{2.33}$$

Indeed

$$\begin{aligned}
e^{-2\eta t}\|\chi_{T+t}\chi_i E^0\|_{L^2}^2 &= \int_{y>a} \chi_{T+t}(y)^2\chi(y)^2 y^{1+2\eta} e^{-2\eta t}\frac{dy}{y^2} \\
&= \int_{r>\ln a} e^{2\eta r}\tilde{\chi}(r-T-t-\ln a)^2\chi(e^r)^2 e^{-2\eta t} dr \\
e^{-2\eta t}\|\chi_{T+t}\chi_i E^0\|_{L^2}^2 &\leqslant \int_{\ln 2a}^{\ln(5a)+T+t} e^{2\eta(r-t)} = \frac{1}{2\eta}[e^{2\eta(\ln(5a)+T)} - e^{2\eta(\ln(2a)-t)}] \\
&\leqslant \frac{C_2}{\eta}(1+o(1)) \\
e^{-2\eta t}\|\chi_{T+t}\chi_i E^0\|_{L^2}^2 &\geqslant \int_{\ln 3a}^{\ln(4a)+T+t} e^{2\eta(r-t)} = \frac{1}{2\eta}[e^{2\eta(\ln(4a)+T)} - e^{2\eta(\ln(3a)-t)}] \\
&\geqslant \frac{C_1}{\eta}(1+o(1)).
\end{aligned}$$

Hence, when $\eta \times t \to +\infty$, and $\lambda > \lambda_{max}$,

$$\begin{aligned}
\eta\langle \mathrm{Op}(\sigma)E_i, E_i\rangle = \eta e^{-2\eta t}\langle\, &\mathrm{Op}(\sigma_{\epsilon,t})\chi_{T+t}\chi_i E^0, \chi_{T+t}\chi_i E^0\rangle \\
&+ \mathcal{O}\left(\frac{e^{-2\eta t}}{\eta} + (h|t|e^{2\lambda|t|})^\infty + h|t|e^{2\lambda|t|}\right).
\end{aligned}$$

Again, this ends the proof, by taking $t = t_0|\log h|/(2\lambda)$ — the above apply because we have $\eta \times |\log h| \to \infty$. □

2.2.2.3. Stationary phase computations

The idea behind the proof here is that $\chi_{T+t}\chi_i E^0$ is a lagrangian state, thus mapped to another lagrangian state by $\mathrm{Op}(\sigma_{\epsilon,t})$ which is a pseudo-differential operator.

Lemma 2.2.12. *Assume t_0, λ and η satisfy the above conditions. Then,*

$$\begin{aligned}
&\eta e^{-2\eta t}\langle \mathrm{Op}(\sigma_{\epsilon,t})\chi_{T+t}\chi_i E^0, \chi_{T+t}\chi_i E^0\rangle = \\
&\qquad \left[2\pi a^{2\eta}\eta e^{-2\eta t}\int d\theta d\tau e^{2\eta\tau}[\chi_{T+t}\chi]^2(ae^\tau)\sigma_{\epsilon,t-\tau}(a,\theta,\frac{1}{a},0)\right] + \mathcal{O}(h^{1-t_0})
\end{aligned}$$

Proof. This computation only takes place in cusp Z_i, and we forget the dependence in i until the end of the proof of this lemma.

First, we can eliminate the integration in the θ' and J variable in the LHS because of the following fact. Let $D \subset \mathbb{R}^d$ be a fundamental domain for Λ. When ς is tempered on $\mathbb{R}^{2d}$, Λ periodic in the first variable,

$$\begin{aligned}
\int_D \left(\int_{\mathbb{R}^{2d}} \varsigma\left(\frac{\theta+\theta'}{2}, hJ\right) e^{i\langle\theta-\theta',J\rangle}d\theta' dJ\right) d\theta &= \int_D \sum_{J\in\Lambda'} \hat{\varsigma}(J/2, hJ/2)e^{i\langle\theta,J\rangle}d\theta \\
&= \int_D \varsigma(\theta,0)d\theta
\end{aligned}$$

where $\hat{\varsigma}$ was the discrete Fourier transform in the first variable. Hence, the quantity in the LHS in the lemma is the integral over $\theta \in D$ of the following expression :

$$h^{-1}\eta e^{-2\eta t}\int y^{s-1}y'^{\bar{s}-1}e^{i(y-y')Y/h}\left\{\chi_{T+t}\chi(y)\right\}\left\{\chi_{T+t}\chi(y')\right\}\sigma_{\epsilon,t}\left(\frac{y+y'}{2},\theta,Y,0\right)dydy'dY. \tag{2.34}$$

We want to use the fact that if ς is a symbol in some $S^n(Z)$, not depending on θ nor on J, then the function $\tilde{\varsigma}(s,v) = \varsigma(e^s, e^{-s}v)$ is a symbol in the usual Kohn-Nirenberg sense, in $S^n(\mathbb{R})$ — notation of definition 2.2.2. Remark that the behavior is not so clear in the θ variable, for which periodicity and rescaling are not compatible.

We introduce the following rescalings : $y = ae^\tau$, $y' = y(1+u)$, $Y = (1+v)/y$. Up to a factor $h^{-1}\eta a^{2\eta}e^{-2\eta(t-\tau)}\chi_{T+t}\chi(ae^\tau)$, the expression in equation (2.34) is the integral over $\tau \in \mathbb{R}$ of

$$\int (1+u)^{\eta-1/2}e^{i(\log(1+u)-u(1+v))/h}\left\{\chi_{T+t}\chi(ae^\tau(1+u))\right\}\sigma_{\epsilon,t}\left(ae^\tau\left(1+\frac{u}{2}\right),\theta,\frac{1+v}{ae^\tau},0\right)dudv.$$

Remark that this integral vanishes when $\tau \notin [\ln 2, T+t+\ln 5]$, and write

$$\sigma_{\epsilon,t}\left(ae^\tau\left(1+\frac{u}{2}\right),\theta,\frac{1+v}{ae^\tau},0\right) = \sigma_{\epsilon,t-\tau}\left(a\left(1+\frac{u}{2}\right),\theta,\frac{1}{a}(1+v),0\right).$$

Then, introduce a cutoff $\varrho(u)$, supported around 0, and $1 = \varrho + 1 - \varrho$ to separate the integral into two parts (I) and (II).

Let us examine first (II) which is not stationary, and supported for $|u| > \delta$. We insert $1 = u^N/u^N$ and integrate by parts in v. We take the L^1 bound, considering that $\sigma_{\epsilon,t}$ is supported in $\{p \in [1/2-\epsilon, 1/2+\epsilon]\}$, and using symbol estimates on $\sigma_{\epsilon,t}$. It gives

$$\begin{aligned}|(\mathrm{II})| \leqslant C_N h^N e^{N\lambda(t-\tau)}\int (1-\varrho(u))(1+u)^{\eta-1/2}\frac{(1+u/2)^N}{u^N}[\chi_{T+t}\chi](ae^\tau(1+u))\\ \mathbb{1}\left[(1+u/2)|1+v| \leqslant \sqrt{1+2\epsilon}\right]dudv.\end{aligned}$$

After some rescaling in the v variable, and considering

$$[\chi_{T+t}\chi](ae^\tau(1+u)) \leqslant \mathbb{1}(-1 \leqslant u \leqslant +\infty),$$

this is bounded (uniformly in θ and τ) by

$$Ch^{N(1-2\rho)}\int_{-1}^{+\infty} du\frac{(1-\varrho(u))(1+u)^{\eta-1/2}\left(1+u/2\right)^{N-1}}{u^N} = \mathcal{O}\left(h^{2-t_0}\right).$$

Part (I) of the integral supported around $u = 0$ is an oscillatory integral that can directly be estimated. Indeed, on that domain, the phase function satisfies symbolic estimates and has only one critical point $(u,v) = (0,0)$, where it is $-uv + \mathcal{O}(u^3, v^3)$. Further consider that the function under the integral is smooth and uniformly compactly supported in v. When we differentiate it in v, we lose a $\mathcal{O}(e^{\lambda|t-\tau|})$ constant. When differentiating in u, either we differentiate $\sigma_{\epsilon,t-\tau}$, losing again a $\mathcal{O}(e^{\lambda|t-\tau|})$ constant, or we differentiate $\rho(u)(1+u)^{\eta-1/2}\chi_{T+t}\chi$. We chose — recall (2.24) — the cutoffs χ_{T+t} and χ exactly so that we lose only $\mathcal{O}(1)$ constants by doing so.

The basic stationary phase theorem — see theorem 7.7.5 in [Hör03] — in the plane applies and we find

$$(\mathrm{I}) = 2\pi h[\chi_{T+t}\chi](ae^{\tau})\sigma_{\epsilon,t-\tau}\left(a,\theta,\frac{1}{a},0\right) + \mathcal{O}(h^{2-t_0}),$$

uniformly in variables θ and τ.

Recall we are to integrate (I)+(II) in θ and τ with a prefactor $h^{-1}\eta a^{2\eta}e^{-2\eta(t-\tau)}\chi_{T+t}\chi(ae^{\tau})$. But

$$\int d\theta d\tau\eta a^{2\eta}e^{-2\eta(t-\tau)}\chi_{T+t}\chi(ae^{\tau}) = \mathcal{O}(1).$$

and that estimate ends the proof. □

2.2.2.4. Dynamical properties and conclusion

Recall from equation (2.23) that whenever ς is a compactly supported continuous function on T^*M, in the coordinates of cusp Z_i,

$$\mu^+_{i,\nu}(\varsigma) = 2\nu a^{2\nu}\int_{t\in\mathbb{R}} e^{-2\nu t}\varsigma\circ\varphi_{-t}\big(a,\theta,\frac{1}{a},0\big)d\theta.$$

When $\eta(h)\to\nu>0$, η remains bounded, and we can directly apply lemma 2.2.12 and equation (2.27). Letting $h\to 0$, we find

$$\eta\langle \mathrm{Op}(\sigma)E_i(s),E_i(s)\rangle \to \pi\mu^+_{i,\nu}(\sigma).$$

Since μ_i^+ is supported on S^*M where σ and σ_ϵ coincide. Actually, when $\eta\to 0$, as slow as the theorem requires, lemma 2.2.12 and equation (2.27) also imply that

$$\begin{aligned}|\eta\langle \mathrm{Op}(\sigma)E_i(s),E_i(s)\rangle - \pi\mu^+_{i,\eta(h)}(\sigma)| &= \mathcal{O}(h^{1-t_0}) + \mathcal{O}(\|\sigma\|_{L^\infty}e^{-2\eta(h)t(h)})\\ &= \mathcal{O}(|\log h|^{-t_0}) \text{ since } 0<t_0<1.\end{aligned}$$

The proof of theorem 2.4 will therefore be complete if we can prove

Lemma 2.2.13. *Assume M has strictly negative curvature. For all $\sigma\in C^0_c(T^*M)$, for all $i=1\dots k$, as $\nu\to 0^+$,*

$$\mu^{\pm}_{i,\nu}(\sigma)\to\int\sigma d\mathscr{L}_1$$

where $\mathscr{L}_1$ is the normalized Liouville measure on the unit cotangent bundle of M.

It is as far as we know an open question as to whether the Liouville measure is a Gibbs measure in such a cusp-manifold — that is to say, whether a Ruelle inequality holds. If it were, we could apply directly theorem 3 in [Bab02]. However, mimicking the proof therein and using the classical Hopf argument, we are able to conclude. Observe that replacing the hypothesis of negative curvature by *ergodic*, or even *mixing*, we are not able to prove that conclusion still holds: we really use the stable and unstable foliations, and the fact that they are absolutely continuous.

In this part of the proof, it is easier to consider only μ^-, that is supported on incoming horospheres.

Proof. From now on, we work on the unit cotangent sphere since both $\mu^-_{\nu,i}$ and $\mathscr{L}_1$ are supported on S^*M.

Take $\varepsilon > 0$. Since σ is compactly supported, we can find a $\delta > 0$ such that $|\sigma(\xi) - \sigma(\xi')| < \varepsilon$ whenever $|\xi - \xi'| < \delta$. The measures $\mu^-_{i,\nu}$ are obtained by propagating an incoming horosphere — by which we mean the set of downwards pointing normal vectors to a horizontal torus in a cusp. Following the idea of proof in [Bab02], we want to thicken the horosphere. Denote by $H_{i,a}$ the incoming horosphere at height a in cusp Z_i, and consider the set

$$\Omega_{i,a,C} := \bigcup_{\xi \in H_{i,a}} B(\xi, C, W^s(\xi)),$$

where $B(\xi, C, W^s(\xi))$ is a ball of radius $C > 0$ in the local weak stable leaf of ξ, centered at ξ. For $C > 0$ small enough, in $\Omega_{i,a,C}$, $H_{i,a}$ is a local section of the weak-stable foliation of S^*M, with projection π^{su}. From the contraction properties on the weak-stable foliation, for some constant $C > 0$, on $\Omega_{i,a,C\delta}$, $|\sigma \circ \varphi_t - \sigma \circ \varphi_t \circ \pi^{su}| \leqslant \varepsilon$ for $t \geqslant 0$.

From theorem 7.6 in [PPS12], there is a locally bounded measurable density ρ on $\Omega_{i,a,C\delta}$ such that we have a ''Fubini" decomposition,

$$d\mathscr{L}_1 = \rho(\xi) d\mathrm{vol}_{W^s(\pi^{su}\xi)} d\mathrm{vol}_{H_{i,a}}.$$

The measure on $H_{i,a}$ being $d\theta$ of mass 1. We let g be the function vanishing outside $\Omega_{i,a,C\delta}$ such that on $\Omega_{i,a,C\delta}$,

$$g(\xi) := \frac{1}{\rho(\xi)\mathrm{vol}(B(\pi^{su}\xi, C, W^s(\pi^{su}\xi)))}.$$

Then, g is in $L^1(S^*M)$, and $\|g\|_{L^1(S^*M)} = \mathrm{vol}(H_{i,a}) = 1$. We have, uniformly for $t \geqslant 0$,

$$\left| \int_{H_{i,a}} \sigma \circ \varphi_t d\theta - \int_{\Omega_{i,a,C\delta}} (\sigma \circ \varphi_t).g d\mathscr{L}_1 \right| \leqslant \varepsilon.$$

Consider that in the definition of $\mu^+_{\nu,i}$, since σ is compactly supported, we can integrate in t for $t \in [-T, +\infty[$ only. Additionally, the prefactor $a^{2\eta}$ tends to 1, and the part $t \in [-T, 0]$ will not contribute, so we write

$$\mu^-_{i,\nu}(\sigma) = \mathcal{O}(\nu T \|\sigma\|_\infty) + 2\nu \int_0^\infty dt e^{-2\nu t} \int \sigma \circ \varphi_t \times g d\mathscr{L}_1 + \mathcal{O}(\varepsilon).$$

Using the Hopf argument (as in [Cou07]), and theorem 7.6 from [PPS12] again, one can see that the geodesic flow is mixing for the Liouville measure. Actually, it suffices for it to be *ergodic*. Indeed,

$$2\nu \int_0^\infty dt e^{-2\nu t} \int \sigma \circ \varphi_t \times g d\mathscr{L}_1 = \int_{\mathbb{R}^+} t e^{-t} \int_{S^*M} g(\xi) F\left(\frac{t}{2\nu}, \xi\right) dt d\mathscr{L}_1(\xi),$$

where $F(t, \xi)$ is the Birkhoff average of σ for a time t along the trajectory of ξ. Since g is L^1, and σ is bounded, by dominated convergence and ergodicity, the limit of this when $\nu \to 0$ is $\|g\|_{L^1}\mathscr{L}_1(\sigma)$. For all $\varepsilon > 0$, we find for any limit value $\overline{\sigma}$ of $\mu^-_{i,\nu}(\sigma)$,

$$|\overline{\sigma} - \mathscr{L}(\sigma)| = \mathcal{O}(\varepsilon).$$

letting $\varepsilon \to 0$ yields the desired result. □

2.3. Complement: A Quantum Ergodicity result, following S.Zelditch

The following theorem is due to S. Zelditch in constant curvature [Zel91].

Theorem 2.5. *Let M be a cusp manifold whose geodesic flow is ergodic. Then, for all compactly supported symbol σ, we have the following convergence, as $h \to 0$.*

$$\begin{aligned} \frac{h^d}{4\pi}\int_{\mathbb{R}} \left| \left\langle \mathrm{Op}(\sigma) E\left(\frac{d}{2}+i\frac{\lambda}{h}\right), E\left(\frac{d}{2}+i\frac{\lambda}{h}\right)\right\rangle + \frac{\varphi'}{\varphi}\left(\frac{d}{2}+i\frac{\lambda}{h}\right)\int_{\lambda S^*M}\sigma \right| d\lambda \\ + h^{d+1}\sum_j \left| \langle \mathrm{Op}(\sigma)u_j, u_j\rangle - \int_{h\lambda_j S^*M}\sigma \right| \to 0. \end{aligned} \tag{2.35}$$

Observe that this is not a variance, but a mean absolute variation. In some sense, it is because the Eisenstein functions are not L^2 that we only get this and not the usual variance.

The rest of this section will contain the proof of this theorem. S. Zelditch had an idea to simplify his own proof in constant curvature, and I worked out the details of a proof based on that idea at the start of my PhD. Later, S. Zelditch shared with me his own version of the proof, which happened to be considerably shorter. This is the proof I follow in the lines below.

There are two parts in the proof. First, we deal with symbols whose average on each energy layer vanishes. This part is very similar to the compact case, and actually uses ergodicity. In the second part, we treat the case of some explicit symbols that are constant in the compact part, so as to be able to use the Maass-Selberg relations.

Let us explain first why that is a reasonable idea. Let $\sigma \in C_c^\infty(T^*M)$. Then, one can define

$$\overline{\sigma}(\lambda) = \int_{|\xi|=\lambda} \sigma d\mathscr{L}_\lambda$$

If the average of $\chi \in C_c^\infty(M)$ is 1 over M, then σ and $\tilde{\sigma} : (x,\xi) \mapsto \chi(x)\overline{\sigma}(|\xi|)$ have the same average in each energy layer. We will first consider the case of $\sigma - \tilde{\sigma}$, whose average is zero. Then, we will treat the case of $\tilde{\sigma}$. We denote by $\varepsilon(\sigma, h)$ the quantity in the LHS of (2.35).

Recall that Π_y^* is the projector on functions whose zero-Fourier mode vanishes for $y_M > y$, and $G^y = \Pi_y^* E$. We are going to use (MS-3). We will also need the following fact:

Lemma 2.3.1. *Let u be a solution of $-h^2\Delta u = u$ that is either a L^2 eigenfunction, or an Eisenstein function. If u_h is a family of such solutions, and if σ is a compactly supported symbol in some S^n_ρ, not supported in $\{p \in [1/2-\epsilon, 1/2+\epsilon]\}$,*

$$\mathrm{Op}(a)u_h = \mathcal{O}(h^\infty)\|u_h\|_{L^2(K)} \tag{2.36}$$

for some compact set K.

Proof. Following the method used after equation (2.29), we take a $\delta > 0$ and we find a symbol r such that $\mathrm{Op}(\sigma) = \mathrm{Op}(r)(P - 1/2) + \mathcal{O}_{y_M^{-\delta}L^2 \to L^2}(h^\infty)$. The space $y_M^{-\delta}L^2$ is the space of functions of the form $y_M^{-\delta} f$ with $f \in L^2(M)$. The Eisenstein function are of this form, as we observed at the start of the proof of lemma 2.2.11. Then, we find

$$\mathrm{Op}(\sigma)u_h = \mathcal{O}(h^\infty)\|u_h\|_{y_M^{-\delta}L^2(M)}. \tag{2.37}$$

Now, the non-zero Fourier modes of u_h are square integrable in y, and the L^2 norm of $\Pi_y^* u_h$ can be controlled uniformly from the L^2 norm of u_h in some compact set. Likewise, the $y_M^{-\delta}L^2$ norm of the zero Fourier mode can be controlled by the L^2 norm of u_h in some compact set, as we can observe from the explicit expression. □

2.3.1. Symbols whose average is zero

Let us start with a symbol σ of average zero, i.e so that for all $\lambda \in \mathbb{R}^+$,

$$\int_{\lambda S^* M} \sigma = 0. \tag{2.38}$$

We also assume that σ has compact support, in $\{p(\xi) < \lambda_0^2\} \cap \{y \leqslant y_0\}$. Then in the formula (2.35), the part of the sum and integral for $|\lambda| \geqslant \lambda_0$ contributes for a $\mathcal{O}(h^\infty)$ remainder.

Indeed, if f is a C^∞ eigenfunction for the eigenvalue $s(d-s)$ with $s = d/2 + i\lambda/h$,

$$f = \frac{1}{h^2 d^2/4 + \lambda^2}(-h^2\Delta)f.$$

Hence,

$$\langle \mathrm{Op}(\sigma)f, f\rangle = \left(\frac{1}{h^2 d^2/4 + \lambda^2}\right)^N \langle \mathrm{Op}(\sigma^N)f, f\rangle,$$

where σ^N is the full symbol of $\mathrm{Op}(\sigma)(-h^2\Delta)^N$. Now, from the localization of eigenfunctions 2.3.1, if $|\lambda| > \lambda_0$, $\langle \mathrm{Op}(\sigma^N)f, f\rangle = \mathcal{O}(h^\infty)\|f\|^2_{y \leqslant y_0}$, uniformly in λ. In particular, the part of $\varepsilon(\sigma, h)$ supported for $|\lambda| > \lambda_0$ is bounded above by

$$\mathcal{O}(h^\infty)\left\{\int_{|\lambda|>\lambda_0}\left(\frac{1}{h^2 d^2/4 + \lambda^2}\right)^N \|G^y\|^2_{y\leqslant y_0} d\lambda + h\sum_{h\lambda_j > \lambda} |\langle \mathrm{Op}(\sigma)u_j, u_j\rangle|\right\}$$

One can recognize the trace of $\Pi_{y_0}^*(-h^2\Delta)^{-N}\mathbb{1}\{-h^2\Delta > h^2d^2/4 + \lambda_0^2\}$. From the lemma 2.2.3, the whole expression is $\mathcal{O}(h^\infty)$.

In what follows, we only consider eigenfunctions whose eigenvalue is smaller than $d^2/4 + \lambda_0^2/h^2$. Let $T > 0$, and take a cutoff $\chi_T \in C_c^\infty(M)$ with value 1 on $\{y \leqslant y_0 e^{2TE}\}$, and supported in $\{y \leqslant e^{3TE}\}$. Then, if $f = u_j$, or if $f = E_i$, with $U(t) = \exp ith\Delta/2$ and $t \in [0, T]$, we find

$$\begin{aligned}
\langle \mathrm{Op}(\sigma)f, f\rangle &= \langle \mathrm{Op}(\sigma)U(t)\chi_T f, U(t)\chi_T f\rangle + \mathcal{O}(h^\infty)\|f\|^2_{L^2(\{y\leqslant y_0 e^{3TE}\})}, \\
&= \langle \mathrm{Op}(\sigma\circ\varphi_t)f, \chi_T f\rangle + \mathcal{O}(h)\|f\|^2_{L^2(\{y\leqslant y_0 e^{3TE}\})}, \\
&= \left\langle \mathrm{Op}\Bigg(\underbrace{\frac{1}{T}\int_0^T \sigma\circ\varphi_t dt}_{:=\eta_T}\Bigg) f, \chi_T f\right\rangle + \mathcal{O}(h)\|f\|^2_{L^2(\{y\leqslant y_0 e^{3TE}\})}.
\end{aligned}$$

To prove this, we use the same ideas as around the equation (2.31). Observe that η_T still is compactly supported. The main ingredients are the localization of eigenfunctions, the localization of $\mathrm{Op}(\sigma)$, and the remark 2.5. Hence, up to a remainder $R(\sigma, h, t)$, we get

$$\varepsilon(\sigma,h) - R(\sigma,h,t) \leqslant h^{d+1} \sum_{h\lambda_j \leqslant E} |\langle \mathrm{Op}(\eta_T)\chi_T u_j, \chi_T u_j\rangle| + \frac{h^d}{4\pi}\sum_i \int_{-\lambda_0}^{\lambda_0} |\langle \mathrm{Op}(\eta_T)E_i, \chi_T E_i\rangle|.$$

Using Cauchy-Schwarz, we can interpret this as the scalar product of the families $\{\mathrm{Op}(\eta_t)f\}$ and $\{\chi_T f\}$ where f runs through the eigenfunctions u_j and E_i. In other words, $\varepsilon(\sigma,h)^2$ is smaller that the product of

$$h^{d+1}\sum_j \|\mathrm{Op}(\eta_T)u_j\|^2 + \frac{h^d}{4\pi}\sum_i \int \|\mathrm{Op}(\eta_T)E_i\|^2, \text{ and } h^{d+1}\sum_{h\lambda_j\leqslant\lambda_0} \|\chi_T u_j\|^2 + \frac{h^d}{4\pi}\sum_i\int_{-\lambda_0}^{\lambda_0} \|\chi_T E_i\|^2.$$

This is the product of the traces of

$$h^{d+1}\,\mathrm{Op}(\eta_T)^*\,\mathrm{Op}(\eta_T) \text{ and } h^{d+1}\mathbb{1}(-h^2\Delta \leqslant h^2 d^2/4 + \lambda_0^2)\chi_T.$$

On the other hand, the remainder $R(\sigma, h, t)$ can be interpreted as the trace of

$$\mathcal{O}(h^{d+2})\mathbb{1}(-h^2\Delta \leqslant h^2 d^2/4 + \lambda_0^2)\chi_T$$

From the semi-classical trace formula in lemma 2.2.3, we find that

$$\varepsilon(\sigma,h)^2 \leqslant \int_{T^*M} |\eta_T|^2 \int \chi_T^2 + \mathcal{O}(h|\log h|).$$

In particular, for all $T > 0$, we find that

$$\limsup_{h\to 0} \varepsilon(\sigma,h)^2 \leqslant |M| \int_{T^*M} |\eta_T|^2.$$

From Von Neumann's ergodic theorem, this tends to 0 as $T \to +\infty$, if the flow is ergodic.

2.3.2. The case of non-zero mean-value.

To conclude the proof, we come back to the case of $\tilde\sigma(x,\xi) = \chi(x)\overline{\sigma}(|\xi|)$. We can stop considering altogether the L^2 eigenfunctions. Indeed, in their contribution to (2.35), we can replace $\overline{\sigma}(h\lambda_j)$ by $\overline{\sigma}(h\lambda_j)\|u_j\|^2$, and we are back to the case $\overline{\sigma} = 0$.

Now, by the localization of eigenfunctions,

$$\left\langle \mathrm{Op}(\tilde\sigma)E\left(\frac{d}{2}+i\frac{\lambda}{h}\right),\ E\left(\frac{d}{2}+i\frac{\lambda}{h}\right)\right\rangle = \int_M \chi\left|E\left(\frac{d}{2}+i\frac{\lambda}{h}\right)\right|^2 \overline{\sigma}\left(\frac{h^2d^2}{4}+\lambda^2\right) + \mathcal{O}(h)\|E\|^2_{\{\chi\neq 0\}}$$

From the observation on $R(\sigma, h, t)$ in the previous arguments, we see that the remainder will again contribute for $\mathcal{O}(h|\log h|)$ in the estimate of the quantity (2.35). We are hence left to study:

$$h^d \int_{-\lambda_0}^{\lambda_0} \overline{\sigma}\left(\frac{h^2 d^2}{4} + \lambda^2\right) \left| \int_M \chi \left| E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 + \frac{\varphi'}{\varphi}\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right| d\lambda$$

Now, we choose the shape χ in the following way. We suppose that χ is constant in M_0, and that in the cusps, it writes as $c_\nu \chi_0(y\nu)$, where $\chi_0 \in C_c^\infty(\mathbb{R})$ equals 1 close to zero, ν is a small parameter and $c_\nu \in \mathbb{R}^+$ is a normalization constant. It is chosen so that χ has mean value 1, i.e $\int_M \chi = \text{vol}(M)$. Write

$$\int_M \chi \left| E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 = -c_\nu \int \nu \chi_0'(y\nu) \left\{ \int_{y_M \leqslant y} |E|^2 \right\} dy.$$

For ν small enough, this is well defined, and we can use the Maass-Selberg formula (MS-3) (this is Zelditch's trick):

$$\int_M \chi \left| E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 = -c_\nu \int \nu \chi_0'(y\nu) \left\{ 2\kappa \log y - \frac{\varphi'}{\varphi} + \text{Tr}\, \frac{y^{2i\lambda/h}\phi^* - y^{-2i\lambda/h}\phi}{2i\lambda/h} \right\} dy$$
$$+ \int_M (1-\chi)\, |\Pi^* E|^2$$

In the RHS, the third term is highly oscillating. Il will only contribute for $\mathcal{O}(h^\infty)$ to the final result, by non-stationary phase. The second will contribute by $-c_\nu \varphi'/\varphi$. The fourth is an integral supported in the cusps, where Π^* is the projection on the non-zero Fourier modes. We are left to prove that

$$h^d \int_{-\lambda_0}^{\lambda_0} \overline{\sigma}\left(\frac{h^2 d^2}{4} + \lambda^2\right) \left\{ \left| 2\kappa \int c_\nu \nu \chi_0'(\nu y). \log y dy \right| + \left| (1 - c_\nu) \frac{\varphi'}{\varphi}\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right| \right\} d\lambda$$
$$+ h^{d+1}\, \text{Tr}\left\{ (1-\chi)\overline{\sigma}(-h^2\Delta)\right\}.$$

goes to 0 with h. Let us observe that when $\nu \to 0$, the assumption that χ has mean value 1 implies that $c_\nu \to 1$.

The first term does not create a problem. The third term is $\mathcal{O}(\text{vol}\{y \geqslant 1/\nu\})$, again from the trace formula 2.2.3.

The second gives

$$h^d |1 - c_\nu| \int_{-\lambda_0}^{\lambda_0} h^d \overline{\sigma}\left(\frac{h^2 d^2}{4} + \lambda^2\right) \left| \frac{\varphi'}{\varphi}\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right| \tag{2.39}$$

In the decomposition (1.32) the sum over the resonances is a function that is always negative for big values of t, and the polynomial part is explicit, so we know that

$$\int_{-\lambda_0}^{\lambda_0} \left| \frac{\varphi'}{\varphi}\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right| + \frac{\varphi'}{\varphi}\left(\frac{d}{2} + i\frac{\lambda}{h}\right) d\lambda = \mathcal{O}(h^{-d}). \tag{2.40}$$

Combining this with the Weyl law (WM2), we arrive at

$$|1-c_\nu| \int_{-\lambda_0}^{\lambda_0} h^d \overline{\sigma} \left(\frac{h^2 d^2}{4} + \lambda^2 \right) \left| \frac{\varphi'}{\varphi} \left(\frac{d}{2} + i \frac{\lambda}{h} \right) \right| \\ = |1-c_\nu| \left\{ \mathcal{O}(1) + \mathcal{O}(h^{d+1})\, 0\text{-Tr}\left(\mathbb{1}\{-h^2\Delta \leqslant \lambda_0\} \right) \right\} = |1-c_\nu| \mathcal{O}(1).$$

At last, we deduce that

$$\limsup_{h\to 0} \varepsilon(\tilde{\sigma}, h) \leqslant o_{\nu\to 0}(1).$$

□

Chapitre 3

Paramétrice pour le déterminant de diffusion et zones sans résonances

Dans ce chapitre, je reproduis le contenu de l'article [Bon15b]. On se donne une variété à pointe M de courbure négative. Il s'agit de construire un noyau de Poisson approximé sur le revêtement universel de M, pour obtenir une paramétrice pour les fonctions d'Eisenstein (Théorème 3.3). Ceci permet de donner une paramétrice pour le déterminant de diffusion, dans un demi-plan $\{\Re s > \delta_g\}$ (Théorème 3.5). L'abscisse de convergence de la paramétrice est reliée à la pression du demi-jacobien instable, ce qui peut rappeler [NZ09]. En utilisant des arguments d'analyse complexe, on trouve des zones sans résonances dans le plan complexe. On obtient aussi un certain nombre d'exemples exceptionnels (Théorème 3.1).

❉

The object of our study are complete $d+1$-dimensional negatively curved manifolds of finite volume (M, g) with a finite number κ of real hyperbolic cusp ends. The Laplace operator is denoted Δ in the analyst's convention that $-\Delta \geqslant 0$. The resolvent $R(s) = (-\Delta - s(d-s))^{-1}$ is a priori defined on $L^2(M)$ for $\Re s > d/2$. Thanks to the analytic structure at infinity, one shows that R can be analytically continued to $\mathbb{C}$ as a meromorphic family of operators $C_c^\infty \to C^\infty$ whose set of poles is called the *resonant set* $\text{Res}(M)$. The original proof is due to Selberg in constant -1 curvature, to Lax and Phillips [LP76] for surfaces, and this subject was studied by both Yves Colin de Verdière [CdV81, CdV83] and Werner Müller [Mül83, Mül86, Mül92]. It fits in the general theory of spectral analysis on geometrically finite manifolds with constant curvature ends, see [MM87, GZ97].

The spectrum of $-\Delta$ divides into both discrete L^2 spectrum, that may be finite, infinite or reduced to $\{0\}$, and continuous spectrum $[d^2/4, +\infty)$. We can find a precise description of the structure of its spectral decomposition given by the Spectral Theorem in [Mül83]. For each cusp Z_i, $i = 1 \dots \kappa$, there is a meromorphic family of *Eisenstein functions* $\{E_i(s)\}_{s\in\mathbb{C}}$ on M such that

$$-\Delta E_i(s) = s(d-s)E_i(s). \tag{3.1}$$

The poles of the family are contained in $\{\Re s < d/2\} \cup (d/2, d]$, and are called *resonances*. We also consider the vector $E = (E_1, \dots, E_\kappa)$. Let $\{u_\ell\}_\ell$ be the discrete L^2 eigenvalues.

Then, any $f \in C_c^\infty(M)$ expands as:

$$f = \sum_\ell \langle u_\ell, f\rangle u_\ell + \frac{1}{4\pi}\sum_i \int_{-\infty}^{+\infty} E_i\left(\frac{d}{2}+it\right)\left\langle E_i\left(\frac{d}{2}+it\right), f\right\rangle dt \quad \text{[Mül83, eq. 7.36]},$$

where $\langle \cdot, \cdot \rangle$ is the L^2 duality product. An important feature of the Eisenstein functions is the following: in cusp Z_j, the zeroth Fourier mode in θ of E_i writes as

$$\delta_{ij} y^s + \phi_{ij}(s) y^{d-s}. \tag{3.2}$$

where ϕ_{ij} is a meromorphic function. If we take the determinant of the *scattering matrix* $\phi = \{\phi_{ij}\}$, we obtain the *scattering determinant* $\varphi(s)$. It is known that the set $\mathcal{R}$ of poles of φ is the same as that of $\{E(s)\}_s$ — again, see [Mül83, theorem 7.24]. It also coincides with the poles of the analytic continuation of the kernel of the resolvent of the Laplacian, [Mül83].

Thanks to the symmetries of the problem, $\varphi(s)\varphi(d-s) = 1$, so that studying the poles of φ in $\{\Re s < d/2\}$ is equivalent to studying the zeroes in $\{\Re s > d/2\}$. In this article, we will be giving information on the zeroes of φ, keeping in mind that the really important objects are the poles.

The first examples of cusp manifolds to be studied had constant curvature, and were arithmetic quotients of the hyperbolic plane. Let $\Gamma_0(N)$ be the congruence subgroup of order N, that is, the kernel of the morphism $\pi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_N)$. Then, $\mathbb{H}/\Gamma_0(N)$ is a *cusp surface*. For such examples, and more generally, for constant curvature cusp surfaces $\mathbb{H}/\Gamma$, if cusp Z_i is associated with the point ∞ in the half plane model, then the associated Eisenstein functions can be written as a series

$$E_i(s)(z) = \sum_{[\gamma]\in\Gamma_i\backslash\Gamma} [\Im(\gamma z)]^s \tag{3.3}$$

where Γ_i is the maximal parabolic subgroup of Γ associated with Z_i. Recall a Dirichlet series is a function of the form

$$f(s) = \sum_{k\geqslant 0} \frac{a_k}{\lambda_k^s}, \text{ where } (\lambda_k) \text{ is a strictly increasing sequence of real numbers.}$$

Selberg proved — see [Sel89b] — that there is a non-zero Dirichlet series L converging absolutely for $\{\Re s > d\}$ so that

$$\varphi(s) = \left(\frac{\pi\Gamma(s-1/2)}{\Gamma(s)}\right)^{\kappa/2} L(s) \tag{3.4}$$

This implies:

Theorem (Selberg)**.** *For constant curvature cusp surfaces, the resonances are contained in a vertical strip of the form* $\{1/2-\delta \leqslant \Re s \leqslant 1/2\}$*, where* $\delta > 0$ *(with maybe the exception of a finite number of resonances in* $(1/2, 1]$*).*

Müller was actually the one who asked if Selberg's theorem still holds in variable curvature. Froese and Zworski [FZ93] gave a counter-example, that had positive curvature. The following theorem gives a partial answer in negative curvature.

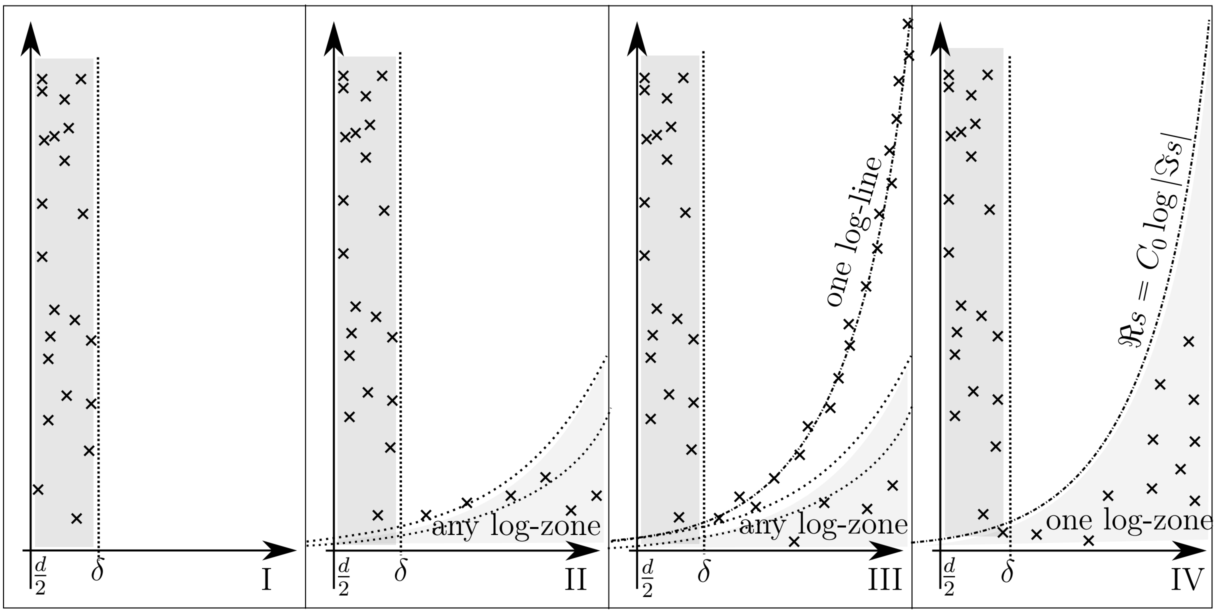

Figure 3.1: The zeroes of φ: 4 cases in theorem 3.1.

Theorem 3.1. *For M a cusp manifold, let $\mathcal{G}(M)$ be the set of C^∞ metrics g on M such that (M, g) is a cusp manifold with negative sectional curvature. If $U \subset\subset M$ is open, let $\mathcal{G}_U(M)$ be the set of metrics in $\mathcal{G}(M)$ that have constant curvature outside of U. Endow $\mathcal{G}(M)$ and $\mathcal{G}_U(M)$ with the C^2 topology on metrics. Then*

(I) *There are hyperbolic cusp surfaces M and non-empty open sets $U \subset\subset M$ such that for all $g \in \mathcal{G}_U(M)$, $\mathrm{Res}(M, g)$ is still contained in a vertical strip.*

(II) *Given any cusp manifold M, for an open and dense set of $g \in \mathcal{G}(M)$, or all of $\mathcal{G}(M)$ when there is only one cusp, there is a $\delta > d/2$ such that for any constant $C > 0$,*
$$\{s \in \mathrm{Res}(M, g),\ \Re s < d - \delta,\ \Re s > -C \log |\Im s|\} \text{ is finite.}$$

(III) *There is a 2-cusped surface (M, g) with the following properties. The resonant set $\mathrm{Res}(M, g)$ is the union of Res_{strip}, Res_{far} and an exceptional set Res_{exc}, so that*
$$\begin{aligned} \mathrm{Res}_{strip} &= \{s \in \mathrm{Res}(M, g),\ \Re s > d - \delta\} \\ \mathrm{Res}_{exc} &= \{s_i, \overline{s_i},\ i \in \mathbb{N},\ s_i = \tilde{s}_i + \mathcal{O}(1)\} \\ \mathrm{Res}_{far} &\cap \{\Re s > -C \log |\Im s|\} \text{ is finite for any } C > 0. \end{aligned}$$
where $\delta > 0$, and the $\tilde{s}_i$'s and $\overline{\tilde{s}_i}$'s are the zeroes of $se^{-sT} - C_0$ for some constants $T > 0$ and $C_0 \neq 0$.

(IV) *For a bigger open and dense set of metrics $g \in \mathcal{G}(M)$, containing the example in (III), there are constants $\delta > 0$, and $C_0 > 0$ such that*
$$\{s \in \mathrm{Res}(M, g),\ \Re s < d - \delta,\ \Re s > -C_0 \log |\Im s|\} \text{ is finite.}$$

Conjecture 3.1. *The set of metrics in* (IV) *is actually* $\mathcal{G}(M)$*.*

Conjecture 3.2. *For an open and dense set of* $g \in \mathcal{G}(M)$*, there is an infinite number of resonances outside of any strip* $d/2 > \Re s > d - \delta$*.*

Our reason for conjecturing this is that the existence of such resonances seems to be more stable than their absence.

The main tool to prove theorem 3.1 is a parametrix for the scattering determinant φ in a half plane $\{\Re s > \delta_g\}$. Thanks to the form of that parametrix — sums of Dirichlet series — we will be able to determine zones where φ does not vanish.

Theorem 3.2. *Let* (M, g) *be a negatively curved cusp manifold. There is a constant* $\delta_g > d/2$ *and Dirichlet series* $L_0, \ldots, L_n, \ldots$ *with abscissa of absolute convergence* δ_g *such that if at least one of the* L_n*'s does not identically vanish, for* $\Re s > \delta_g$*, as* $\Im s \to \pm\infty$*,*

$$\varphi(s) \sim s^{-\kappa d/2} L_0 + s^{-\kappa d/2-1} L_1 + \ldots.$$

(Recall, κ *is the number of cusps). Actually, the constant* δ_g *is the pressure of the potential* $(F^{su} + d)/2$*, where* F^{su} *is the unstable jacobian. In constant curvature,* $F^{su} = -d$ *and* $\delta_g = d$*.*

This is a consequence of a more precise estimate — see Theorem 3.5. The L_n's are defined by dynamical quantities related to *scattered geodesics*. Those are geodesics that come from one cusp and escape also in a cusp — maybe the same — spending only a finite time in the compact part of M, called the *Sojourn Time*. This terminology was introduced by Victor Guillemin [Gui77]. In that article, for the case of constant curvature, he gave a version similar to ours of (3.4). He also conjectured that something along the lines of our theorem should hold — see the concluding remarks pp. 79 in [Gui77]. Lizhen Ji and Maciej Zworski gave a related result in the case of locally symmetric spaces [JZ01].

Sojourn Times are objects in the general theory of classical scattering — see [PS03]. Maybe ideas from different scattering situations may help to prove Conjecture 3.1, that may be reformulated as

Conjecture 1'. *Given* $g \in \mathcal{G}(M)$*, at least one* L_i *is not identically zero.*

The structure of the article is the following. In section 3.1 we recall some definitions and results on cusp manifolds, and prove the convergence of a modified Poincaré series. Section 3.2 is devoted to building a parametrix for the Eisenstein functions, via a WKB argument, using the modified Poincaré series. In section 3.3, we turn to a parametrix for the scattering determinant. To use Stationary Phase, most of the effort goes into proving the non-degeneracy of a phase function. The purpose of section 3.4 is to study the behaviour of the series L_i when we vary the metric. Finally, we prove theorem 3.1 in section 3.5. In appendix A.1, for lack of a reference, we give a proof of a regularity result on horocycles. This result may be of interest for the study of negatively curved geometrically finite manifolds in general.

This work is part of the author's PhD thesis. In a forecoming article [Bon15a], we will deduce precise spectral counting results from Theorem 3.2.

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3.1. Scattered geodesics and some potential theory on cusp manifolds

Recall that a manifold N is said to have *bounded geometry* when its injectivity radius is strictly positive, and when $\nabla^k R$ is bounded for all $k = 0, 1, \ldots,$ R being the Riemann curvature tensor of N. Since the injectivity radius goes to zero in a cusp, a cusp manifold cannot have bounded geometry. However, its universal cover $\widetilde{M}$ does. Since the curvature of M is negative, $\widetilde{M}$ is also a Hadamard space — diffeomorphic to $\mathbb{R}^{d+1}$ — and we can define its visual boundary $\partial_\infty \widetilde{M}$, and visual compactification $\overline{M} = \widetilde{M} \cup \partial_\infty \widetilde{M}$.

In all the article, unless stated otherwise, we will refer to the projection $T^*\widetilde{M} \to \widetilde{M}$ as π; when we say *geodesic*, we always mean *unit speed* geodesic.

The results given without proof are from the book [PPS12].

3.1.1. Hadamard spaces with bounded geometry and negative curvature

Let us define the *Busemann cocycle* in the following way. For $p \in \partial_\infty \widetilde{M}$, Let

$$\beta_p(x, x') := \lim_{w \to p} d(x, w) - d(x', w).$$

For each $p \in \partial_\infty \widetilde{M}$, we pick $m_p \in \widetilde{M}$ — we will specify this choice later, see remark 3.2. Then, we define the horosphere $H(p, r)$ (resp. the horoball $B(p, r)$) of radius $r \in \mathbb{R}$ based at p as

$$H(p,r) := \left\{ x \in \widetilde{M} \;\middle|\; \beta_p(x, m_p) = -\log r \right\} \text{ and } B(p,r) := \left\{ x \in \widetilde{M} \;\middle|\; \beta_p(x, m_p) \leqslant -\log r \right\}. \tag{3.5}$$

We also define

$$G_p(x) := \beta_p(x, m_p). \tag{3.6}$$

Beware that with these notations, horoballs $B(p, r)$ increase in size as r decreases. The number r will correspond to a height y in the coming developments.

Since the curvature of $\widetilde{M}$ is pinched-negative $-k_{max}^2 \leqslant K \leqslant -k_{min}^2$, $\widetilde{M}$ has the *Anosov property*. That is, at every point of $S^*\widetilde{M}$, there are subbundles such that

$$T(S^*M) = \mathbb{R}\mathbf{X} \oplus E^s \oplus E^u$$

where $\mathbf{X}$ is the vector field of the *geodesic flow* φ_t. This decomposition is invariant under φ_t, and there are constants $C > 0, \lambda > 0$ such that for $t > 0$

$$\|d\varphi_t|_{E^s}\| \leqslant Ce^{-\lambda t} \text{ and } \|d\varphi_{-t}|_{E^u}\| \leqslant Ce^{-\lambda t}.$$

The subbundle E^s (resp. E^u) is tangent to the *strong stable* (resp. *unstable*) foliation W^s (resp. W^u). The subbundles E^s, E^u are only Hölder — see [PPS12, theorem 7.3] — but each leaf of W^s, W^u is a $\mathscr{C}^\infty$ submanifold of $\widetilde{M}$ — see lemma A.1.1.

Remark 3.1. *We have to say how we measure regularity on $\widetilde{M}$ and $T\widetilde{M}$. In $TT\widetilde{M}$, we have the* vertical *subbundle $V = \ker T\pi : TT\widetilde{M} \to T\widetilde{M}$. Since $\widetilde{M}$ is riemannian, we* *also have a horizontal subbundle H given by the connection ∇. Both V and H can be identified with $T\widetilde{M}$, and the Sasaki metric is the one metric on $T\widetilde{M}$ so that $V \perp H$ and those identifications are isometries.*

We endow $T\widetilde{M}$ with the Sasaki metric, and then also $T^*\widetilde{M}$ by requesting that $v \mapsto \langle v, \cdot \rangle$ is an isometry. For a detailed account on the Sasaki metric, see [GK02]. On all the manifolds that appear, when they have a metric, we define their $\mathscr{C}^k$ spaces, $k \in \mathbb{N}$, using the norm of their covariant derivatives:

$$\|f\|_{\mathscr{C}^n} := \sup_{k=0,\dots,n} \|\nabla^k f\|_\infty.$$

Then, $\mathscr{C}^\infty = \cap_{n \geqslant 0} \mathscr{C}^n$. For a more detailed account of $\mathscr{C}^k$ spaces on a riemannian manifold, see for example the appendix "functionnal spaces in a cusp" in [Bon14a].

There are useful coordinates for describing the geodesic flow φ_t on $S^*\widetilde{M}$. We associate its endpoints p^-, p^+ with a geodesic. Then we have the identification

$$S^*\widetilde{M} \simeq \partial^2_\infty \widetilde{M} \times \mathbb{R}$$

given by $\xi \mapsto (p^-, p^+, t = \beta_{p^-}(\pi\xi, m_{p^-}))$. Here $\partial^2_\infty \widetilde{M}$ is obtained by removing the diagonal from $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$. In those coordinates, φ_t is just the translation by t in the last variable. Moreover, the *strong unstable manifold* of ξ is the set $\{p^- = p^-(\xi), t = t(\xi)\}$. For the strong stable manifold, it is a bit more complicated in this choice of coordinates.

We deduce that $W^u(\xi)$ is the set outer normal bundle to the horosphere based at $p^-(\xi)$, through $\pi\xi$. The horospheres $H(p,r)$ are $\mathscr{C}^\infty$ submanifolds, and each G_p is a smooth function so that $dG_p \in \mathscr{C}^\infty(\widetilde{M})$. The proof uses the fact that the unstable manifolds W^u are $\mathscr{C}^\infty$ (lemma A.1.1), and the fact that there can be no conjugate points in negative curvature.

For $p \in \partial_\infty \widetilde{M}$, we introduce $W^{u0}(p)$ as the set of $\xi \in S^*\widetilde{M}$ such that $p^-(\xi) = p$. It is the set of outer normals to horospheres based at p. It is the graph of dG_p, and

$$G_p(\pi\varphi_t(x, \mathrm{d}G_p)) = G_p + t.$$

We will refer to $W^{u0}(p)$ as the *incoming Lagrangian* from p.

3.1.2. Parabolic points and scattered geodesics

Now, let $\Gamma = \pi_1(M)$. It is a discrete group acting freely on $\widetilde{M}$ by isometries. The elements of Γ can be seen to act by homeomorphisms on $\overline{M}$. We can define the limit set $\Lambda(\Gamma)$ as $\overline{\Gamma \cdot x^0} \cap \partial_\infty \widetilde{M}$, where the closure was taken in $\overline{M}$, and x^0 is an arbitrary point in $\widetilde{M}$. This does not depend on x^0.

If $\gamma \in \Gamma$ is not the identity, one can prove that it has either. (1) Exactly one fixed point in $\widetilde{M}$, (2) Exactly two fixed points on $\partial_\infty \widetilde{M}$, (3) Exactly one fixed point in $\partial_\infty \widetilde{M}$. Then we say that it is (1) elliptic, (2) loxodromic, or (3) parabolic. Here there are no elliptic elements in Γ, since Γ acts freely on $\widetilde{M}$. Our study will be focused of the parabolic elements of Γ.

All the parabolic elements γ of Γ are *regular*, in the following sense: there is $r_\gamma \in \mathbb{R}^*_+$ so that if p_γ is the fixed point of γ, $B(p_\gamma, r_\gamma)$ has constant curvature -1. We denote by

Γ_{par} the set of parabolic elements in Γ. The set of p_γ's is the set of *parabolic points* of $\partial_\infty \widetilde{M}$, Λ_{par}.

Let $p \in \Lambda_{par}$. Then, horoballs centered at p will project down to M as neighbourhoods of some cusp Z_i, and we say that p is a parabolic point that *represents* Z_i. When $\gamma.p = p$, we also say that γ represents Z_i. Objects (points in the boundary, or elements of Γ_{par}) representing the same cusp will be called equivalent. Γ acts on Γ_{par} by conjugation, and elements of the same orbit under Γ are equivalent — however observe that the equivalence classes gather many different orbits under Γ.

If p is a parabolic point representing Z_i, write $p \in \Lambda^i_{par}$. Let $\Gamma_p < \Gamma$ be its stabilizer. It is a *maximal parabolic subgroup*. We always have $\Gamma_p \simeq \pi_1(\mathbb{T}^d_i) \simeq \mathbb{Z}^d$. The set of parabolic points equivalent to p is in bijection with $\Gamma_p \backslash \Gamma = \{\Gamma_p\gamma,\ \gamma \in \Gamma\}$.

The following lemma seems to be well known in the literature. However, since we cannot give a reference for a proof, we have written one down.

Lemma 3.1.1. *Since M has finite volume, $\Lambda(\Gamma)$ is the whole boundary, and the parabolic points are dense in $\partial_\infty \widetilde{M}$.*

Proof. Let us pick a cusp Z_i, and a point $p \in \Lambda^i_{par}$. Then, we consider $x \in \partial Z_i$ in the boundary of Z_i in M. We can lift x to $\tilde{x} \in H(p, a_i)$. The orbit under Γ of any $\tilde{x}' \in H(p, a_i)$ will remain at bounded distance of the orbit of $\tilde{x}$ under Γ. We deduce that $\Lambda(\Gamma)$ is the intersection of the closure of $\cup_\gamma \gamma H(p, a_i)$ with the boundary $\partial_\infty \widetilde{M}$. This implies in particular that $\Lambda^i_{par} \subset \Lambda(\Gamma)$.

Now, we can find a distance d on $\overline{M}$ that is compatible with its topology. Indeed, take a point $m \in \widetilde{M}$, and consider the distance $\tilde{d}$ obtained on $\overline{M}$ by requesting that

$$v \in B(0,1) \subset T\widetilde{M} \mapsto \exp_z \{v \times \operatorname{argth}|v|\} \text{ is an isometry.}$$

Then, for that distance, the sequence of images $\gamma H(p, a_i)$ have shrinking radii. Now, take a sequence of points $\tilde{x}_j \in \gamma_j H(p, a_i)$, so that $\tilde{x}_j \to q \in \Lambda(\Gamma)$. We have $\gamma_j H(p, a_i) = H(\gamma_j p, a_i)$, and so $\tilde{d}(\tilde{x}_j, \gamma_j p) \to 0$. This proves that $\Lambda(\Gamma) = \overline{\Lambda^i_{par}}$.

Next, consider the open set U in $\widetilde{M}$ obtained by taking only points of $\widetilde{M}$ that project to points in the compact part $\mathring{M}_0 \subset\subset M$. There is $C > 0$ such that given $\tilde{x}_1 \in U$, for any $\tilde{x}_2 \in U$, there is a $\gamma \in \Gamma$ such that $d(\gamma\tilde{x}_1, \tilde{x}_2) \leqslant C$.

Let $\overline{U}$ be the closure of U in $\overline{M}$. Since U is at distance at most C of the orbit of any of its points under Γ, we deduce that the limit set is $\overline{U} \cap \partial_\infty \widetilde{M}$.

Then, we find that $\Lambda(\Gamma) = \overline{\cup_\gamma \gamma H(p, a_i)} \cap \partial_\infty \widetilde{M} = \overline{U} \cap \partial_\infty \widetilde{M}$. But, we also have $\overline{\cup_\gamma \gamma H(p, a_i)} \cap \partial_\infty \widetilde{M} = \overline{\cup_\gamma \gamma B(p, a_i)} \cap \partial_\infty \widetilde{M}$. We deduce that

$$\Lambda(\Gamma) = \left\{ \overline{\cup_\gamma \gamma B(p, a_i)} \cup \overline{U} \right\} \cap \partial_\infty \widetilde{M} = \overline{U \cup_\gamma \gamma B(p, a_i)} \cap \partial_\infty \widetilde{M} = \partial_\infty \widetilde{M}. \tag{3.7}$$

□

Remark 3.2. *We will not use the functions G_p when p is not a parabolic point. When $p \in \Lambda^i_{par}$, one can choose the point m_p so that G_p coincides with $-\log \tilde{y}_p$ on the horoball $H(p, r_p)$, where $\tilde{y}_p$ is obtained on $H(p, r_p)$ by lifting the height function y on the cusp Z_i. With this choice, for $p \in \Lambda_{par}$ and $\gamma \in \Gamma$, we have the equivariance relation*

$$G_{\gamma^{-1}p} = G_p \circ \gamma. \tag{3.8}$$

Geodesics that enter a cusp eventually come back to M_0 when they are not *vertical*, that is, when they are not directed along $\pm\partial_y$. A geodesic that *is* vertical in a cusp is said to escape in that cusp.

Definition 3.1.2. *The* scattered geodesics *are geodesics on M that escape in a cusp for both $t \to +\infty$ and $t \to -\infty$.*

The set of scattered geodesics is denoted by $\mathcal{SG}$. Such a geodesic, when lifted to $\widetilde{M}$, goes from one parabolic point to another, and hence is entirely determined by its endpoints. Take p, q representing Z_i, Z_j. For γ, $\gamma' \in \Gamma$, the pair of endpoints $(p, \gamma q)$ and $(\gamma' p, \gamma'\gamma q)$ represent the same geodesic on M. We let $\mathcal{SG}_{ij}$ be the set of geodesics scattered from Z_i to Z_j. From the above, we deduce that when $i \neq j$,

$$\mathcal{SG}_{ij} \simeq \Gamma_i \backslash \Gamma / \Gamma_j \quad \text{and} \quad \mathcal{SG}_{ii} \simeq \Gamma_i \backslash (\Gamma - \Gamma_i)/\Gamma_i, \tag{3.9}$$

where Γ_i (resp. Γ_j) is any maximal parabolic subgroup representing Z_i (resp Z_j).

On the other hand, we can consider the set of C^1 curves that start in Z_i above the torus $\{y = a_i\}$ and end in Z_j, above the torus $\{y = a_j\}$. Among those curves, we can consider the classes of equivalence under free homotopy. Let $\pi_1^{ij}(M)$ be the set of such classes. One can prove that in each class $[c] \in \pi_1^{ij}(M)$, there is exactly one element $\bar{c}$ of $\mathcal{SG}_{ij}$. In particular, this proves that $\mathcal{SG}$ is countable. Hence, we have an identification $\mathcal{SG}ij \simeq \pi_1^{ij}(M)$. In what follows, when there is no ambiguity on the metric, we will write directly $c \in \pi_1^{ij}(M)$. In section 3.4, we will study variations of the metric, and will come back to the notation $[\bar{c}] \in \pi_1^{ij}(M)$.

For a scattered geodesic c_{ij}, we define its *Sojourn Time* in the following way. Take one of its lifts $\tilde{c}_{ij}$ to $\widetilde{M}$, with endpoints p, q. Let T be the (algebraic) time that elapses between the first time $\tilde{c}_{ij}$ hits $\{\tilde{y}_p = a_i\}$, and the last time it crosses $\{\tilde{y}_q = a_j\}$. Then, let

$$\mathcal{T}(c_{ij}) := T - \log a_i - \log a_j. \tag{3.10}$$

This does not depend on the choice of a_i and a_j (as defined in (1.2)), nor on the choice of the lift $\tilde{c}_{ij}$. We say that $\mathcal{T}(c_{ij})$ is the *Sojourn Time* of c_{ij}, and we can see $\mathcal{T}$ as a function on $\pi_1^{ij}(M)$. Given $T > 0$, there is a finite number of $c \in \mathcal{SG}_{ij}$ with sojourn time less than T (otherwise, we would have two such curves that would be so close from one another that they would be homotopic).

We denote by $\mathcal{ST}$ (resp. $\mathcal{ST}_{ij}$) the set of $\mathcal{T}(c)$ for scattered geodesics (resp. between Z_i and Z_j). We also call the *Sojourn Cycles* and denote by $\mathcal{SC}$ the set of sums

$$\mathcal{T}_1 + \cdots + \mathcal{T}_\kappa \tag{3.11}$$

where σ is a permutation of $\{1, \ldots, \kappa\}$, and $\mathcal{T}_i \in \mathcal{ST}_{i,\sigma(i)}$. A set of scattered geodesics $\{c_1, \ldots, c_\kappa\}$ such that $c_i \in \mathcal{SG}_{i\sigma(i)}$ will be called a *geodesic cycle.*

3.1.3. A convergence lemma for modified Poincaré series

Poincaré series is a classical object of study in the geometry of negatively curved spaces — see [DOP00] for example. For Γ a group of isometries on $\widetilde{M}$, its Poincaré series at $x \in \widetilde{M}$ is

$$P_\Gamma(x, s) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma x)}, \quad s \in \mathbb{R}.$$

More generally, given a *Potential* on $S\widetilde{M}$, i.e a Hölder function V on $S\widetilde{M}$ invariant by Γ, its *Poincaré series* is

$$P_{\Gamma,V}(x,s) := \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x} V-s}$$

where $\int_x^{\gamma x} V-s$ is the integral of $V-s$ along the geodesic from x to γx. The convergence of both series does not depend on x, only on s.

Remark 3.3. *We will not write $\int(V-s)$ to reduce the size of the expressions. We will assume that the integrand is all that is written after the sign $\int$, until we encounter another $\int$ sign.*

When p is a point on the boundary, x and x' in $\widetilde{M}$, $\int_x^p - \int_{x'}^p V$ will refer to the limit of $\int_x^{\tilde p} V - \int_{x'}^{\tilde p} V$ as $\widetilde{M} \ni \tilde p \to p$. When V is Hölder, this limit exists because the geodesics $[x,p]$ and $[x',p]$ are exponentially close.

When we sum over $\{[\gamma] \in \Xi\backslash\Gamma\}$ we mean that we sum over a set of representatives for $\Xi\backslash\Gamma$ (Ξ being assumed to be a subgroup of Γ).

We only work with reversible potentials V. That means that $\imath V$ is cohomologous to V (following [PPS12], $\imath$ is the antipodal map in $S\widetilde{M}$). In other words, we require that

$$\int_x^y V - \imath V = A(y) - A(x) \tag{3.12}$$

where A is a bounded Hölder function on $S\widetilde{M}$, invariant by Γ. In particular when this is the case, we can replace V by $\imath V$ in the integrals, losing a $\mathcal{O}(1)$ remainder. It is then harmless to integrate along a geodesic in a direction or the other.

In our case where Γ is the π_1 of M, it is a general fact that there is a finite $\delta(\Gamma,V)\in\mathbb{R}$ such that $P_{\Gamma,V}$ converges for $s > \delta(\Gamma,V)$ and diverges for $s < \delta(\Gamma,V)$. This number is called the *critical exponent* of (Γ,V). We also call $\delta_\Gamma = \delta(\Gamma,0)$ the critical exponent of Γ.

Remark 3.4. *The exponent of convergence of a maximal parabolic subgroup Γ_p is always $\delta_{\Gamma_p} = d/2$. Additionally, the Poincaré series for Γ_p diverges at $d/2$ (Γ_p is* divergent*). This can be seen computing explicitely with the formula for the distance between two points (y,θ) and (y,θ') in the half-space model of the real hyperbolic space $\mathbb{H}^{d+1}$*

$$d((y,\theta),(y,\theta)) = 2\mathrm{argsh}\frac{|\theta-\theta'|}{2y}. \tag{3.13}$$

Definition 3.1.3. *In what follows, we say that a potential V is* admissible *if the following holds. First, V is Hölder function on $S\widetilde{M}$, invariant by Γ and reversible. Second, there are positive constants C,λ, and a constant $V_\infty\in\mathbb{R}$ such that whenever $T>0$, if $\pi\varphi_t(\xi)$ stays in an open set of constant curvature -1 for $t\in[0,T]$, then for $t\in[0,T]$,*

$$|V(\varphi_t(\xi)) - V_\infty| \leqslant Ce^{-\lambda t}. \tag{3.14}$$

Observe that an admissible potential has to be bounded. We will mostly use the potential $V_0 = (F^{su}+d)/2$ where F^{su} is the unstable jacobian (see (3.30) and (3.31)). We start with the following lemma:

Lemma 3.1.4. *Let V be an admissible potential. Then $\delta(\Gamma,V) > \delta_{\Gamma_p} + V_\infty$.*

If $V = 0$, this is the consequence of [DOP00, Proposition 2]. We will actually follow their proof closely, but before, we need 2 observations on triangles in $\widetilde{M}$.

Remark 3.5. ***1.*** *Consider a triangle with sides a, b, c and angles α, β, γ in a complete Hadamard space $\widetilde{M}_k$ of curvature $-k^2$. We have*

$$\cosh kc = \cosh ka \cosh kb - \sinh ka \sinh kb \cos \gamma.$$

Assume that $\gamma > \pi/2$ (the triangle is obtuse). Then we find that there is a constant $C_k > 0$ — smooth in $k \neq 0$ and $k \neq \infty$ — such that

$$|c - (a + b)| \leqslant C_k. \tag{3.15}$$

Since the curvature of M is pinched, by the Topogonov comparison theorem for triangles, the same is true for obtuse triangles in $\widetilde{M}$, with a constant C controlled by k_{min} and k_{max}.

2. *Now, we consider a triangle with sides c_0, c_1, c_2 in $\widetilde{M}$, and V an admissible potential on $\widetilde{M}$. Take $C > 0$. Among those triangles, we restrict ourselves to the ones such that the length of c_0 is at most C. Then*

$$\int_{c_1} - \int_{c_2} V = \mathcal{O}(1). \tag{3.16}$$

this is still valid if the vertex at $c_1 \cap c_2$ is at infinity. Actually, to prove this, first observe that it suffices to make computations for that case when $c_1 \cap c_2$ is at infinity. Then it follows directly from the fact that the two curves are exponentially close in that case.

Proof of lemma 3.1.4. The limit set of Γ_p is reduced to $\{p\}$. In $H(p, a_i)$, Γ_p has a Borelian fundamental $\mathscr{B}$ domain whose closure is compact. We can obtain a fundamental domain $\mathscr{G}$ for Γ_p on $\partial_\infty \widetilde{M} \backslash \{p\}$ by taking the positive endpoints of geodesics from p through $\mathscr{B}$. From [PPS12, Proposition 3.9], which is due to Patterson, there exists a Patterson density μ of dimension $\delta(\Gamma, V)$ on $\widetilde{M}$, i.e, a family of finite non-zero borelian measures $(\mu_x)_{x \in \widetilde{M}}$ on $\partial_\infty \widetilde{M}$, so that for any $x, x' \in \widetilde{M}$, $\gamma \in \Gamma$,

$$\gamma_* \mu_x = \mu_{\gamma x} \quad \frac{d\mu_x}{d\mu_{x'}}(q) = \exp\left\{ \int_x^q - \int_{x'}^q V - \delta(\Gamma, V) \right\}, \ q \in \partial_\infty \widetilde{M}. \tag{3.17}$$

Additionally, the μ_x's are exactly supported on $\Lambda(\Gamma) = \partial_\infty \widetilde{M}$, so $\mu_x(\mathscr{G}) > 0$. Take $x \in \mathscr{B}$. We have

$$\infty > \mu_x(\partial_\infty \widetilde{M}) = \sum_{\gamma \in \Gamma_p} \mu_x(\gamma \mathscr{G}) + \mu_x(\{p\})$$

But,

$$\mu_x(\gamma^{-1} \mathscr{G}) = \gamma_* \mu_x(\mathscr{G}) = \int_{\mathscr{G}} \exp\left\{ \int_{\gamma x}^q - \int_x^q V - \delta(\Gamma, V) \right\} d\mu_x(q)$$

So we find

$$\int_{\mathscr{G}} \sum_{\gamma \in \Gamma_p} \exp\left\{ \int_{\gamma x}^q - \int_x^q V - \delta(\Gamma, V) \right\} d\mu_x(q) < \infty.$$

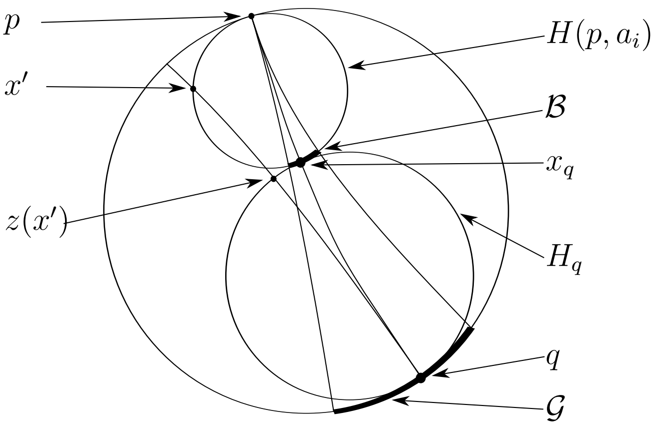

Figure 3.2:

For $q \in \mathscr{G}$, let $x_q \in \mathscr{B}$ be its projection on $H(p, a_i)$. Since we have $d(x, x_q) = \mathcal{O}(1)$ — from the choice of $\mathscr{B}$ — we use (3.16) and uniformly in $\gamma \in \Gamma_p$,

$$\int_{x_q}^{q} - \int_{x}^{q} V - \delta(\Gamma, V) = \mathcal{O}(1) \quad ; \quad \int_{x_q}^{\gamma x} - \int_{x}^{\gamma x} V - \delta(\Gamma, V) = \mathcal{O}(1)$$

Take $z(x')$ the intersection of the geodesic $[q, x']$ and the horosphere H_q based at q through x_q. The set of $z(x')$, $x' \in H(p, a_i)$ has to be bounded. Indeed, $H(p, a_i)$ is not compact, but the only way to go to infinity in $H(p, a_i)$ is to tend to p, and we find that as $x' \to p$, $z(x') \to x_q$. The geometry is described in figure 3.2.

Using again (3.16),

$$\int_{x_q}^{q} - \int_{z(x')}^{q} V - \delta(\Gamma, V) = \mathcal{O}(1) \quad ; \quad \int_{x'}^{z(x')} - \int_{x'}^{x_q} V - \delta(\Gamma, V) = \mathcal{O}(1),$$

and sum everything up (with $x' = \gamma x$)

$$\begin{aligned} \int_{\gamma x}^{q} - \int_{x}^{q} V - \delta(\Gamma, V) &= \mathcal{O}(1) + \int_{\gamma x}^{z(\gamma x)} + \int_{z(\gamma x)}^{q} - \int_{x_q}^{q} V - \delta(\Gamma, V) \\ &= \mathcal{O}(1) + \int_{\gamma x}^{x_q} V - \delta(\Gamma, V) \\ &= \mathcal{O}(1) + \int_{\gamma x}^{x} V - \delta(\Gamma, V). \end{aligned} \tag{3.18}$$

As a consequence,

$$P_{\Gamma_p, V}(x, \delta(\Gamma, V))\mu_x(\mathscr{G}) < \infty,$$

and since $\mu_x(\mathscr{G}) > 0$,

$$P_{\Gamma_p, V}(x, \delta(\Gamma, V)) < \infty. \tag{3.19}$$

Since V is an admissible potential, $P_{\Gamma_p,V}(x, \delta(\Gamma, V)) \asymp P_{\Gamma_p}(x, \delta(\Gamma, V) - V_\infty)$. Since Γ_p is divergent, we deduce that $\delta(\Gamma, V) - V_\infty > d/2$. $\square$

In the article, we will need the convergence of a *modified Poincaré series*. Take V an admissible potential. For a cusp Z_i, take a point $p \in \Lambda^i_{par}$, and let $\pi_p^{a_i}$ be the intersection of the geodesic through p and x with $H(p, a_i)$. The horoballs $B(p, a_i)$, $p \in \Lambda^i_{par}$ are all pairwise disjoint. Indeed, the restriction of the projection $\widetilde{M} \to M$ to any such horoball is a universal cover of Z_i. This implies that for $x \in B(p, a_i)$, the part of the orbit of x under Γ that stays in $B(p, a_i)$ has to be its orbit under Γ_p.

For $x \in M$, take $\tilde{x} \in \widetilde{M}$ a lift of x, and define

$$P_{Z_i,V}(x, s) := \sum_{[\gamma]\in\Gamma_p\backslash\Gamma, \gamma\tilde{x}\notin B(p,a_i)} \exp\left\{ \int_{\pi_p^{a_i}(\gamma\tilde{x})}^{\gamma\tilde{x}} V - s \right\}.$$

This does not depend on the choice of $\tilde{x}$. Given a point $x \in \widetilde{M}$, among a family $\{\gamma x, [\gamma] \in \Gamma_p\backslash\Gamma\}$, there is at most one point in $B(p, a_i)$, and such a point has to be one that minimizes G_p. So, for a point $x \in M$, let x_p be a lift minimizing G_p among the lifts of x. For $s \in \mathbb{R}$, also let $G_i(x) := G_p(x_p)$. For $q \in \Lambda^j_{par}$, also let

$$P_V^{ij}(s) := \sum_{\Gamma_p\gamma\Gamma_q\neq\Gamma_p} \exp\left\{ (V_\infty - s)\mathcal{T}(p, \gamma q) + \int_p^{\gamma q} V - V_\infty \right\}, \tag{3.20}$$

where $\mathcal{T}(p, \gamma q)$ is the sojourn time of the geodesic on M that lifts to $[p, \gamma q]$. Observe that the set $\{\Gamma_p\gamma\Gamma_q \neq \Gamma_p\}$ can be identified with $\mathcal{SG}_{ij}$, from equation (3.9). The main result of this section is

Lemma 3.1.5. *The series $P_{Z_i,V}(x, s)$ and P_V^{ij} converge if and only if $s > \delta(\Gamma, V)$. Additionally, when $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that for $s > \delta(\Gamma, V) + \epsilon$,*

$$\|P_{Z_i,V}(x, s)\|_{L^2(M)} \leqslant C_\epsilon \tag{3.21}$$

Our proof is inspired by [BHP01], and we generalize their Theorem 1.1. One can also see the article [PP13], or the proposition 3 in [Pau13]. For two real valued functions f and g, we write $f \asymp g$ when there is a constant $C > 0$ with $Cg \leqslant f \leqslant g/C$. In the following, when we use that notation, we let the constant C depend on s, but not on x, γ, p. We fix a cusp Z_i, a representing parabolic point $p \in \Lambda^i_{par}$.

Proof. The proof is divided into 3 parts. First, we compare the values of terms of the sum for different x's, to check that the convergence does not depend on x indeed. Then, we study the sum for some well chosen x, to find the convergence exponent. At last, we turn to asymptotics in cusps. We let P^* be the series where we have not excluded $\gamma x_q \in B(p, a_i)$ from the sum.

1. Take x, x' two points in M, at distance $D > 0$, and two lifts $\tilde{x}$ and $\tilde{x}'$ such that $d(\tilde{x}, \tilde{x}') = D$.

Take $\gamma \in \Gamma$. Assume that $G_p(\gamma\tilde{x}') \geqslant G_p(\gamma\tilde{x})$. Then the projection x^1_γ of $\gamma\tilde{x}'$ on the horoball $B(p, G_p(\gamma\tilde{x}))$ is at distance $\mathcal{O}(D + 1)$ from $\gamma\tilde{x}$. This is a simple consequence of equation (3.15) for the triangle with vertices $\gamma\tilde{x}, \gamma\tilde{x}', x^1_\gamma$. Write

$$\int_{\pi_p^{a_i}(\gamma\tilde{x}')}^{\gamma\tilde{x}'} V - s - \int_{\pi_p^{a_i}(\gamma\tilde{x})}^{\gamma\tilde{x}} V - s = \int_{x^1_\gamma}^{\gamma\tilde{x}'} V - s + \int_{\pi_p^{a_i}(\gamma\tilde{x}')}^{x^1_\gamma} V - \int_{\pi_p^{a_i}(\gamma\tilde{x})}^{\gamma\tilde{x}} V.$$

Since V is Hölder, and bounded, we deduce that

$$\int_{\pi_p^{a_i}(\gamma\tilde{x}')}^{\gamma\tilde{x}'} V - s - \int_{\pi_p^{a_i}(\gamma\tilde{x})}^{\gamma\tilde{x}} V - s = \mathcal{O}(D+1)\left\{(1+|s|) + \int_0^\infty (e^{-k_{min}t})^\mu dt\right\}$$

where μ is the Hölder exponent of V. The constants in the estimates do not depend on x and x'. We have used that the geodesics joining $\gamma\tilde{x}$, $\pi_p^{a_i}\gamma\tilde{x}$ and x_γ^1, $\pi_p^{a_i}\gamma\tilde{x}'$ are on the same strong stable manifold. We deduce that for some constant $C > 0$,

$$e^{-C(D|s|+1)} \leqslant \frac{P^*(x', s)}{P^*(x, s)} \leqslant e^{C(D|s|+1)} \quad x, x' \in M, d(x, x') = D. \tag{3.22}$$

2. Take now a point $x \in M$ so that $x_p \in H(p, a_i) \subset B(p, a_i)$. We claim that for all $x' \in H(p, a_i)$,

$$\int_{x'}^{\gamma x_p} V - s = (V_\infty - s)d(x', \pi_p^{a_i}(\gamma x_p)) + \mathcal{O}(1) + \int_{\pi_p^{a_i}(\gamma x_p)}^{\gamma x_p} V - s. \tag{3.23}$$

The remainder being bounded independently from x_p and γ. Let us assume that this holds for now. Then, we write

$$\begin{aligned} P_{\Gamma,V}(x_p, s) - P_{\Gamma_p,V}(x_p, s) &= \sum_{\Gamma_p\gamma\neq\Gamma_p}\sum_{\alpha\in\Gamma_p} \exp\left\{\int_{\alpha x_p}^{\gamma x_p} V - s\right\}, \\ &\asymp \sum_{\Gamma_p\gamma\neq\Gamma_p}\sum_{\alpha\in\Gamma_p} \exp\left\{(V_\infty - s)d(\alpha x_p, \pi_p^{a_i}(\gamma x_p)) + \int_{\pi_p^{a_i}(\gamma x_p)}^{\gamma x_p} V - s\right\} \\ &\asymp P_{Z_i,V}(x_p, s)P_{\Gamma_p}(x_p, s - V_\infty). \end{aligned}$$

Hence

$$P_{Z_i,V}(x_p, s) \asymp \frac{P_{\Gamma,V}(x_p, s) - P_{\Gamma_p,V}(x_p, s)}{P_{\Gamma_p}(x_p, s - V_\infty)}.$$

But from lemma 3.1.4, we know that $\delta(\Gamma, V) > \delta(\Gamma_p, V)$.

The proof of 3.23 is left as an exercise, very similar to the proof of 3.18 — replacing q by γx_p, γx by $\pi_p^{a_i}(\gamma x_p)$ and x by x'.

3. We turn to asymptotics in the cusps. Take $x \in Z_j$ and $q \in \Lambda_{par}^j$ (if $i = j$, take $p = q$). Let x_q minimize G_q among the lifts of x. Observe that the map $(\Gamma_p\gamma\Gamma_q \neq \Gamma_p, \alpha \in \Gamma_q) \mapsto \Gamma_p\gamma\alpha$ is a bijection onto $\Gamma_p\backslash\Gamma$ if $i \neq j$, and $\Gamma_p\backslash(\Gamma - \Gamma_p)$ if $i = j$. We hence rewrite

$$P_{Z_i,V}(x, s) = \sum_{\Gamma_p\gamma\Gamma_q\neq\Gamma_p}\sum_{\alpha\in\Gamma_q} \exp\left\{\int_{\pi_p^{a_i}(\gamma\alpha x_q)}^{\gamma\alpha x_q} V - s\right\}.$$

Consider H_q the horosphere based at q, through x_q. Let z_γ (resp. z'_γ) be the point of intersection of the geodesic $[p, \gamma q]$ with γH_q (resp. $H(p, a_i)$). From (3.18), we have

$$\int_{\pi_p^{a_i}(\gamma\alpha x_q)}^{\gamma\alpha x_q} V - s = \mathcal{O}(1) + \int_p^{z_\gamma} + \int_{z_\gamma}^{\gamma\alpha x_q} - \int_p^{\pi_p^{a_i}(\gamma\alpha x_q)} V - s.$$

However, the distance between z'_γ and $\pi_p^{a_i}(\gamma\alpha x_q)$ is uniformly bounded. This is a direct consequence of lemma 3.3.2. Hence

$$\int_{z'_\gamma}^{p} - \int_{\pi_p^{a_i}(\gamma\alpha x_q)}^{p} = \mathcal{O}(1),$$

and

$$\int_{\pi_p^{a_i}(\gamma\alpha x_q)}^{\gamma\alpha x_q} V - s = \left\{ \int_p^{\gamma q} V - V_\infty \right\} + (V_\infty - s)\left(\mathcal{T}(p,\gamma q) - G_q(x_q) + d(\gamma^{-1} z_\gamma, \alpha x_q)\right) + \mathcal{O}(1).$$

where $\mathcal{T}(p,\gamma q)$ is the sojourn time for the geodesic $[p,\gamma q]$. It follows that

$$P_{Z_i,V}(x,s) \asymp \sum_{\Gamma_p\gamma\Gamma_q\neq\Gamma_p} \exp\left\{(V_\infty - s)\mathcal{T}(p,\gamma q) + \int_p^{\gamma q} V - V_\infty\right\} \times \sum_{\alpha\in\Gamma_q} \exp\left\{(V_\infty - s)(-G_q(x_q) + d(\gamma^{-1} z_\gamma, \alpha x_q))\right\}$$

In the RHS, the first term does not depend on x; we recognize $P_V^{ij}(s)$. The second is related to $P_{\Gamma_q}(x_q)$. We can see it as a Riemann sum as $x_q \to q$. Indeed, $\Gamma_q \simeq \mathbb{Z}^d$, and we can write explicitely the second term as

$$e^{(s-V_\infty)G_q(x_q)} \sum_{\theta\in\Lambda_i} \exp\left\{2(V_\infty - s)\operatorname{argsh}\frac{|\theta-\theta_0|}{2e^{-G_q(x_q)}}\right\} \tag{3.24}$$

As $x_q \to q$, $y = e^{-G_q(x_q)} \to +\infty$, and we can see this as a Riemann sum for the function $f = \exp\{2(V_\infty - s)\operatorname{argsh}\}$ for the parameter $2y$. It should be equivalent to $(2y)^d \int_{\mathbb{R}^d} f$. However f is integrable if and only if $s - V_\infty > d/2$. As a result, we find that

$$\sum_{\alpha\in\Gamma_q} \exp\left\{(V_\infty - s)(-G_q(x_q) + d(\gamma^{-1} z_\gamma, \alpha x_q))\right\} \asymp e^{(s-V_\infty-d)G_q(x_q)}, \quad s > V_\infty + d/2.$$

It is easy to check that the L^2 norm of this is finite whenever $s \geqslant V_\infty + d/2 + \epsilon$. The proof of the lemma is complete when we observe that the L^2 norm decreases when $\Re s$ increases. □

3.2. Parametrix for the Eisenstein functions

In the case of constant curvature, the universal cover $\widetilde{M}$ is the real hyperbolic space $\mathbb{H}^{d+1}$. On it, there is the *Poisson kernel* $P(x,p,s)$ that associates a function on the boundary $f(p)$ with a function on $\mathbb{H}^{d+1}$, $u(x)$ such that

$$(-\Delta - s(d-s))u(x) = 0 \quad u(x) = \int P(x,p,s)f(p)dp.$$

In addition, we require that u corresponds to the superposition of outgoing stationary plane waves at frequency s, with weight $f(p)$ in the direction p. When the curvature is variable, one cannot build such a kernel anymore, because the geometry of the space near the boundary is quite singular. In other words, the metric structure on the boundary is not differentiable, only Hölder. Hence, no satisfactory theory of distributions is available. However, in the special case of parabolic points that correspond to hyperbolic cusps, the fact that small enough horoballs have constant curvature enables us to construct an *approximate* Poisson kernel for $p \in \Lambda_{par}$.

Taking the half space model for $\mathbb{H}^{d+1}$, the Poisson kernel for the point $p = \infty$ is $P = y^s$, so one can rewrite formula (3.3) as

$$E_i(s,x) = \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} P(\gamma x, p, s).$$

This is exactly the type of expression we are looking for. In the first subsection, we introduce some notations. In the second we recall some facts on Jacobi fields that we will need. Then we build the approximate Poisson kernel, and later, we prove that summing over $p \in \Lambda^i_{par}$ gives a good approximation of E_i.

3.2.1. Some more notations

Fix some Z_i and let $p \in \Lambda^i_{par}$ be a parabolic point. We denote by φ^p_t the flow on $\widetilde{M}$ generated by ∇G_p. It is conjugated to the geodesic flow on $W^{u0}(p)$ by the projection $\pi : T^*\widetilde{M} \to \widetilde{M}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Jac} \varphi^p_t|_{t=0} = \operatorname{Tr} \nabla^2 G_p = \Delta G_p,$$

so that the Jacobian is

$$\operatorname{Jac} \varphi^p_t = \exp\left\{\int_0^t \Delta G_p \circ \varphi^p_\tau \mathrm{d}\tau\right\}. \tag{3.25}$$

Thanks to the rigid description in the cusps, we have

$$G_p \leqslant -\log a_i \Leftrightarrow \text{ we are above cusp } Z_i \text{ and } G_p = -\log y_i, \text{ for all } p \in \Lambda^i_{par}.$$

In that case, we can compute $\Delta G_p = d$, and it makes sense to define a *twisted* Jacobian:

$$\tilde{J}_p(x) := \lim_{t\to+\infty} \sqrt{\operatorname{Jac} \varphi^p_{-t} e^{td}} = \sqrt{\operatorname{Jac} \varphi^p_{-t} e^{td}}_{t\geqslant G_p(x)+\log a_i}, \text{ for } p \in \Lambda^i_{par}. \tag{3.26}$$

This $\tilde{J}_p$ is constant equal to 1 in the horoball $B(p, a_i)$. It is useful to define

$$b_i := \inf\{y > 0,\ B(p, y) \text{ has constant curvature}\}. \tag{3.27}$$

We have $b_i \leqslant a_i$, and $\tilde{J}$ equals 1 on $B(p, b_i)$. We also let

$$F_p(x) := \log \tilde{J}_p(x). \tag{3.28}$$

Recall the curvature of M is pinched between $-k^2_{max} \leqslant -1 \leqslant -k^2_{min} < 0$. Then by Rauch's comparison theorem, [CE08, Theorem 1.28],

$$d(1 - k_{max}) \leqslant \frac{2F_p}{(G_p + \log b_i)^+} \leqslant d(1 - k_{min}). \tag{3.29}$$

What is more, by A.5.1, $\nabla^n F_p$ is bounded for $n \geqslant 1$, because ∇G_p is in $\mathscr{C}^\infty(\widetilde{M})$.

On the other hand, the Unstable Jacobian F^{su} is the Hölder function on SM defined by

$$F^{su}(x,v) := -\frac{d}{dt}_{|t=0} \det\left[(d\varphi_t)_{|E^u(x,v)}\right] < 0. \tag{3.30}$$

The fact that it is Hölder is a consequence of the Hölder regularity of E^u — see [PPS12, Theorem 7.1]. In what follows, we will be interested by the potential

$$V_0 = \frac{1}{2}F^{su} + \frac{d}{2}. \tag{3.31}$$

We let $\delta_g = \delta(\Gamma, V_0)$. This is the relevant abscissa of convergence of theorem 3.2 in the introduction, as we will see.

3.2.2. Unstable Jacobi fields

We want to relate V_0 and F_p. We have to make a digression, and recall some facts on Jacobi fields. Take a geodesic $x(t)$, and a Jacobi field J along $x(t)$, orthogonal to $x'(t)$. By parallel transport, one can reduce J to some function of time valued in $T_{x(0)}M$. If one also uses parallel transport for the curvature tensor, we get the equation

$$J''(t) + K(t)J(t) = 0. \tag{3.32}$$

if $x(t)$ lives in constant curvature -1, K is the constant matrix $-\mathbf{1}$. If $J(0) = J'(0)$, then $J(t) = e^t J(0)$, and conversely, if $J(0) = -J'(0)$, $J(t) = e^{-t}J(0)$.

For $v \in TM$, denote by $v^\perp$ the space of vectors in T_xM orthogonal to v. Recall that H and V are the horizontal and vertical subspaces introduced in remark 3.1. Then we can identify $T_vSM \simeq (\mathbb{R}v \oplus v^\perp) \oplus v^\perp$. In this identification, the first term $\mathbb{R}v \oplus v^\perp$ is H. The second term $v^\perp$ is $V \cap T_vSM$. In this notation, $\mathbb{R}v$ is the direction of the geodesic flow, and v its vector.

This identification is consistent with Jacobi fields in the sense that if

$$\mathrm{d}\varphi_t.(l, v_1, v_2) = (l(t), v_1(t), v_2(t)),$$

then $l(t) = l$ for all t, $v_1(t)$ is a Jacobi field orthogonal to $v(t) = x'(t)$, and $v_2(t)$ is its covariant derivative (also orthogonal to $v(t)$).

An *unstable Jacobi field* $\mathbb{J}^u(t)$ along $x(t)$ is a $d \times d$ matrix-valued solution of 3.32 along $x(t)$ that is invertible for all time, and that goes to 0 as $t \to -\infty$ — it just gathers a basis of solutions. Similarly, one can define the *stable* Jacobi fields. Such fields always exist; they never vanish, nor does their covariant derivative — see [Rug07]. We denote by $\mathbb{J}^u_t(s)$ the unstable Jacobi field that equals $\mathbf{1}$ for $s = t$ — given a geodesic $x(t)$. Actually, $s \mapsto \mathbb{J}^u_t(t+s)$ only depends on $v = (x(t), x'(t)) \in SM$. We will write it $s \mapsto \mathbb{J}^u_v(s)$.

From the identification with TSM, we find that vectors in E^u take the form $(\mathbb{J}^u(t)w, \mathbb{J}^{u\prime}(t)w)$, whence we deduce that

$$E^u_v = \{(w, \mathbb{J}^{u\prime}_v(0)w) | w \perp v\}. \tag{3.33}$$

The matrix $\mathbb{J}_v^{u\prime}(0)$ only depends on v, we denote it by $\mathbb{U}_v$. Similarly, we define $\mathbb{S}_v$ for the *stable* Jacobi fields. They satisfy the Ricatti equation (along a geodesic $v(t)$):

$$\mathbb{U}' + \mathbb{U}^2 + K = 0.$$

They take values in symmetric matrices (with respect to the metric), which is equivalent to saying that the stable and unstable directions are Lagrangians. Given a geodesic curve $x(t)$, $\mathbb{J}_u(t)$ and $\mathbb{J}_s(t)$ two Jacobi fields along it, we can write $\mathbb{U} = (\mathbb{J}_u^{-1})^T(\mathbb{J}_u')^T$, and find that

$$\frac{d}{dt}\left\{\mathbb{J}_u^T(\mathbb{U}-\mathbb{S})\mathbb{J}_s\right\} = 0. \tag{3.34}$$

This is a Wronskian identity. We can also compute

$$\det d\varphi_{t|E^u(v)} = \det \mathbb{J}_v^u(t)\sqrt{\frac{\det\left(\mathbf{1}+\mathbb{U}^2_{\varphi_t(v)}\right)}{\det\left(\mathbf{1}+\mathbb{U}_v^2\right)}}. \tag{3.35}$$

We have a map $i^u : w \in H(v) \mapsto (w, \mathbb{U}_v w) \in E^u(v)$ from the horizontal subspace to the unstable one. If one considers the metric ds_u^2 obtained on E^u by restriction of the Sasaki metric in TSM, this gives a structure of Euclidean bundle to E^u over SM.

Lemma 3.2.1. *The matrix $\mathbf{1}+\mathbb{U}_v^2$ is the matrix of the metric $i^*ds_u^2$ on H. This is bounded uniformly on SM.*

Proof. this metric is always $\geqslant \mathbf{1}$ — here, $\mathbf{1}$ refers to the metric on H, i.e, the metric on TM. The only way it can blow up would be that for a sequence of v, $\tilde{v} \perp v$, $\mathbb{U}_v\tilde{v} \to \infty$. If v_∞ was a point of accumulation of v in $\widetilde{M}$, that implies that E^u and H are not transverse at v_∞. That is not possible since there are no conjugate points in strictly negative curvature. We deduce that $\pi v \in M$ has to escape in a cusp.

However, in the cusp, the curvature K is constant with value $-\mathbf{1}$. Hence, unstable Jacobi fields in the cusp write as $Ae^t + Be^{-t}$, where A and B are constant matrices along the orbit. Then $\mathbb{U}_v = \mathbf{1} + \mathcal{O}(e^{-t})$ as the point v travels along a trajectory φ_t that remains in a cusp. In particular, $i^*ds_u^2 = 2.\mathbf{1} + \mathcal{O}(1/y)$ for points of height y in a cusp. □

In this context, from the definition, we find that for $x \in \widetilde{M}$,

$$\tilde{J}_p^2(x) = e^{td}\det \mathbb{J}^u_{(x,\nabla G_p(x))}(-t), \text{ for } t \geqslant G_p(x) + \log a_i. \tag{3.36}$$

As a consequence,

Lemma 3.2.2. *For $x \in \widetilde{M}$, and $t \in \mathbb{R}$,*

$$\int_x^{\varphi_t^p(x)} V_0 = F_p(\varphi_t^p(x)) - F_p(x) + \mathcal{O}(1).$$

What is more, V_0 is an admissible potential.

Proof. The first part of the lemma comes directly from equations (3.36) and (3.35), and the observation just afterward.

To prove the second part, it suffices to prove that F^{su} is an admissible potential. Consider a point $v \in TSM$ so that $\varphi_t(v)$ remains in a cusp for times $t \in [0, T]$. Taking the Jacobi fields starting from v along its orbit, for $t \in [0, T]$, we find

$$\mathbb{U}_{\varphi_t v} = (Ae^t - Be^{-t})(Ae^t + Be^{-t})^{-1} = \mathbf{1} + \mathcal{O}(e^{-t}), \tag{3.37}$$

and

$$F^{su} = -\frac{d}{ds}_{|s=0} \left\{ \det \mathbb{J}^u_{\varphi_t(v)}(s) \sqrt{\frac{\det \mathbf{1} + \mathcal{O}(e^{-t})}{\det \mathbf{1} + \mathcal{O}(e^{-t-s})}} \right\} = -d + \mathcal{O}(e^{-t}).$$

The last thing we have to check is that F^{su} is reversible. However, $\imath F^{su}$ is the strong *Stable* Jacobian F^{ss}

$$F^{ss} = \frac{d}{dt}_{|t=0} \log \det d\varphi_{t|E^s(x,v)}. \tag{3.38}$$

From equation (3.35), and the Wronskian identity (3.34), we find that

$$\det d\varphi_{t|E^s(x,v)} \det d\varphi_{t|E^u(x,v)} = \frac{\det \mathbb{U}_v - \mathbb{S}_v}{\det \mathbb{U}_{\varphi_t(v)} - \mathbb{S}_{\varphi_t(v)}} \sqrt{\frac{\det\left(1 + \mathbb{U}^2_{\varphi_t(v)}\right)\left(1 + \mathbb{S}^2_{\varphi_t(v)}\right)}{\det\left(1 + \mathbb{U}^2_v\right)\left(1 + \mathbb{S}^2_v\right)}}. \tag{3.39}$$

Since the function

$$\frac{\sqrt{\det(1 + \mathbb{U}^2)(1 + \mathbb{S}^2)}}{\det \mathbb{U} - \mathbb{S}} \tag{3.40}$$

is well defined on $S\widetilde{M}$, Hölder continuous, and bounded, F^{su} is reversible. □

3.2.3. On the universal cover

In this section, we fix $p \in \Lambda_{par}$, and we omit the dependency on p; it shall be restored afterwards. We use notations introduced in section 3.2.1. We will use the WKB Ansatz to find our approximate Poisson kernel. Consider a formal series of functions on $\widetilde{M}$,

$$f(x) = \sum_{n \geqslant 0} s^{-n} f_n,$$

with $s \in \mathbb{C}$ and $f_0 = 1$, and compute

$$(-\Delta - s(d - s))[e^{-sG}\tilde{J}f] = e^{-sG}\left[s\left(2\nabla G.\nabla(\tilde{J}f) + \tilde{J}f\Delta G - \tilde{J}fd\right) - \Delta(\tilde{J}f)\right],$$

where we have used that G satisfies the eikonal equation $|\nabla G|^2 = 1$. If we expand the formal series, we find that this expression (formally) vanishes if for all $k > 0$,

$$2\tilde{J}\nabla G.\nabla f_n = \Delta(\tilde{J}f_{n-1}).$$

Indeed,

$$2\nabla G.\nabla \tilde{J} = \tilde{J}(d - \Delta G).$$

We can rewrite those equations in terms of $F = \log \tilde{J}$:

$$2\nabla G.\nabla f_n = Qf_{n-1} \text{ where } Qf(x) = \Delta f + 2\nabla F.\nabla f + (|\nabla F|^2 + \Delta F)f. \tag{3.41}$$

These are transport equations, with solutions :

$$f_n = \frac{1}{2}\int_{-\infty}^{0} (Qf_{n-1}) \circ \varphi_\tau^p \mathrm{d}\tau \tag{3.42}$$

Remark that on $\{G \leqslant -\log b\}$, from the definition (3.27) F vanishes, and so does Qf_0. Hence all f_n's but f_0 vanish, and the formula above is legit. We prove :

Lemma 3.2.3. *There are constants $C_{n,N} > 0$ for $n > 1$, $N \in \mathbb{N}$, such that for all $\tau \in \mathbb{R}^+$*

$$\|f_n\|_{\mathscr{C}^N(\{G_p \leqslant \tau - \log b\})} \leqslant C_{n,N}\tau^n.$$

Proof. We use lemma A.5.1 again, and proceed by induction. The result is obvious for $n = 0$. Now assume it holds for some $n \geqslant 0$. Taking $g_0 = f_n$, $g_1 = f_{n+1}$, $\ell = -\log b$, the lemma enables us to conclude directly if we can prove that

$$\|Qf_n\|_{\mathscr{C}^k(G \leqslant \tau - \log b)} \leqslant C_{n,k}\tau^n.$$

But this is a simple consequence of the induction hypothesis and the fact that $\nabla F \in \mathscr{C}^\infty(\widetilde{M})$. □

All the functions defined above depended on choosing a parabolic point p, and now we make it appear in the notations:

$$f_p^N(s) := \sum_{n=0}^{N} s^{-n} f_{n,p} \text{ and } P_N(\cdot, p, s) := e^{-sG}\tilde{J}_p f_p^N(s). \tag{3.43}$$

This is the approximate Poisson kernel. Then for all $N > 0$,

$$[-\Delta - s(d-s)]e^{-sG_p}\tilde{J}_p f_p^N(s) = -s^{-N}e^{-sG_p}\tilde{J}_p Q_p f_{N,p}.$$

so we let

$$R_N(., p, s) := -e^{-sG_p}\tilde{J}_p Q_p f_{N,p} \tag{3.44}$$

This will be the remainder term. Now, as the last point in this section, observe the equivariance relation

$$P_N(\gamma x, p, s) = P_N(x, \gamma^{-1}p, s). \tag{3.45}$$

3.2.4. Poincaré series and convergence

The functions defined by (3.43) and (3.44) in $\widetilde{M}$ are already invariant under the action of Γ_p, so to define a function on M, we only have to sum over $\Gamma_p\backslash\Gamma$. As in section 1.3, take a cusp Z_i, a parabolic point $p \in \Lambda_{par}^i$. For $x \in M$, let $x_p \in \widetilde{M}$ be a point minimizing G_p amongst the lifts of x. Then

Lemma 3.2.4. *For $\epsilon > 0$, and $N \in \mathbb{N}$, there is a constant $C_{N,\epsilon} > 0$ such that for all $x \in M$, and all $\Re s > \delta(\Gamma, V_0) + \epsilon$,*

$$\left\| \sum_{[\gamma]\in\Gamma_p\backslash\Gamma, [\gamma]\neq[0]} |P_N(\gamma x_p, p, s)| \right\|_{L^2_x(M)} < C_{N,\epsilon} a_i^{\Re s}.$$

Further, with the same condition on s, the remainder satisfies

$$\left\| \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} |R_N(\gamma x_p, p, s)| \right\|_{L^2_x(M)} < C_{N,\epsilon} b_i^{\Re s}.$$

with b_i as defined in (3.27).

Proof. First, we give a proof for $N = 0$. write

$$\sum_{[\gamma]\in\Gamma_p\backslash\Gamma,[\gamma]\neq[0]} |P_0(\gamma x_p, p, s)| = \sum_{[\gamma]\in\Gamma_p\backslash\Gamma,[\gamma]\neq[0]} \exp\left\{ \Re s \log a_i + F_p(\gamma x_p) - \int_{\pi_p^{a_i}(\gamma x_p)}^{\gamma x_p} \Re s \right\}.$$

Recall $-F_p(\pi_p^{a_i}(\gamma x_p)) = 0$. By lemma 3.2.2, losing constants not depending on s, the RHS is comparable with

$$a_i^{\Re s} \sum_{[\gamma]\in\Gamma_p\backslash\Gamma,[\gamma]\neq[0]} \exp\left(\int_{\pi_p^{a_i}(\gamma x_p)}^{\gamma x_p} V_0 - \Re s \right).$$

Lemma 3.1.5 states that the term in the right part of the product is bounded uniformly in L^2 norm.

Now, we deal with the higher order of approximation. Let $n > 0$ and consider the sum

$$\sum_{[\gamma]\in\Gamma_p\backslash\Gamma} (e^{-sG_p} \tilde{J}_p f_{n,p}) \circ \gamma.$$

By lemma 3.2.3, this is bounded term by term by

$$C_k \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} (e^{-sG_p} \tilde{J}_p((G_p + \log b_i)^+)^n) \circ \gamma.$$

Inserting $1 = b_i^s b_i^{-s}$, this is

$$C_k b_i^s \sum_{[\gamma]\in\Gamma_q\backslash\Gamma, G_q\geqslant -\log b_i} \partial_s^k (e^{-s(G_q+\log b_i)} \tilde{J}_q) \circ \gamma. \tag{3.46}$$

Let

$$L_0^{b_i} := \sum_{[\gamma]\in\Gamma_q\backslash\Gamma, G_q\geqslant -\log b_i} (e^{-s(G_q+\log b_i)} \tilde{J}_q) \circ \gamma.$$

By the argument above, for $s > \delta(\Gamma, V_0) + \epsilon$, this sum converges and the value is bounded in L^2 norm by some function of s. What is more, since all the exponents are nonpositive, this a decreasing function of $s \in \mathbb{R}$. We deduce that when $\epsilon > 0$, there is $C_\epsilon > 0$ such that for $x \in M$, $\Re s > \delta(\Gamma, V_0) + \epsilon$, we have $\|L_0^{b_i}\|_{L^2} \leqslant C_\epsilon$.

Consider $L = \sum a_k \lambda_k^s$ a Dirichlet series, with $a_k \in \mathbb{R}^+$, $\lambda_k \geqslant 1$, converging for $\Re s > s_0$. Then, if $s - \epsilon > s_0$, we find $|L'(s)| \leqslant L(\Re s - \epsilon) \sup_n |\log \lambda_n| \lambda_n^{-\epsilon}$. Since $L_0^{b_i}$ has this Dirichlet series structure in the s variable, it implies that for some constants $C_{\epsilon,k} > 0$,

$$\|\partial_s^k L_0^{b_i}(s)\|_{L^2} \leqslant C_{\epsilon,k}, \;\; x \in M, \;\; \Re s > \delta(\Gamma, V_0) + \epsilon.$$

Observe that $C_{\epsilon,k}$ may depend on b_i. Hence

$$\left\| \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} |(e^{-sG_p}\tilde{J}_p f_{n,p})\circ\gamma| \right\|_{L^2(M)} \leqslant C_{\epsilon,n} b_i^{\Re s},\ x \in M,\ \Re s > \delta(\Gamma, V_0) + \epsilon.$$

Moreover, this also holds if we replace $f_{n,p}$ by $Q_p f_{N,p}$, and this observation concludes the proof. □

Now, we can prove our first theorem:

Theorem 3.3 (Parametrix for the Eisenstein functions)**.** *For $N \in \mathbb{N}$, let Z_i be some cusp, and $p \in \Lambda^i_{par}$ a representing point. For $\Re s > \delta(\Gamma, V_0)$, let*

$$E_{i,N}(s,x) := \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} P_N(\gamma x_p, p, s).$$

this function is defined on $\widetilde{M}$, but invariant by Γ, so it descends to M; it does not depend on the choice of $p \in \Lambda^i_{par}$. Then, uniformly in s when $\Re s$ stays away from $\delta(\Gamma, V_0)$, and $s \notin [d/2, d]$,

$$\|\partial_s^m (E_i - E_{i,N})\|_{H^k(M)} = \mathcal{O}\left(s^{k-N} b_i^s\right) \tag{3.47}$$

Proof. From lemma 3.2.4, we deduce that $E_{i,N}$ is well defined; it does not depend on p thanks to the equivariance relation (3.45). Additionally, for a cutoff χ that equals 1 sufficiently high in Z_i and vanishes outside of Z_i, $E_{i,N} - \chi y_i^s$ is in L^2, uniformly bounded in sets $\{\Re s > \delta(\Gamma, V_0) + \epsilon\}$. The sum

$$\sum_{[\gamma]\in\Gamma_p\backslash\Gamma} R_N(\gamma\cdot, p, s)$$

converges normally on compact sets, and in $L^2(M)$ also, so we find

$$E_i - E_{i,N} = s^{-N}\left(-\Delta - s(d-s)\right)^{-1} \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} R_N(\gamma\cdot, p, s)$$

Since $\partial_s^m\left(-\Delta - s(d-s)\right)^{-1}$ is bounded on $H^n(M)$ with norm $\mathcal{O}(1)$ when s stays in sets $\{\Re s > d/2 + \epsilon, s \notin [d/2, d]\}$, it suffices to prove that when $\Re s > \delta(\Gamma, V_0) + \epsilon$,

$$\left\| \partial_s^m \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} R_N(\gamma\cdot, p, s) \right\|_{H^k(M)} = \mathcal{O}(s^k b_i^{\Re s})$$

Actually, since the sum has a Dirichlet series structure, we see that this is true for all $m \geqslant 0$ as long as it is true for $m = 0$. From the bounds in lemma 3.2.3, and the bounds on $\nabla G_p \in \mathscr{C}^\infty$, we see that for $x' \in \widetilde{M}$,

$$\|\nabla^k R_N(\cdot, p, s)\|(x') \leqslant C e^{-sG_p(x')}\tilde{J}_p((G_p + \log b_i)^+)^N$$

We conclude the proof using the arguments of the proof of lemma 3.2.4 again — from equation (3.46) and below. □

Remark 3.6. *We have given estimates for the convergence in H^k, $k \geqslant 0$. However, the sum also converges normally in $\mathscr{C}^k$ topology on compact sets.*

3.3. Parametrix for the scattering matrix

Let us recall that the zero-Fourier mode of E_i at cusp Z_j is

$$y^s\delta_{ij} + \phi_{ij}(s)y^{d-s}.$$

This formula is valid a priori for $y \geqslant a_j$. However, if we integrate E_i along a projected horosphere of height $b_i \leqslant y \leqslant a_i$, we still obtain the same expression, even though the projected horosphere may have self-intersection — recall they are the projection in M of horospheres in $\widetilde{M}$. This is true because following those projected horospheres, we do not leave an open set of constant curvature -1 — see (3.27) — and we can apply a unique continuation argument.

The smaller the b_i's are, the better the remainder is. In constant curvature, there is no remainder — the remainder in 3.47 goes to zero as $N \to \infty$, with fixed s. Observe that the parameters b_i are only related to the support of the variations of the curvature, and not to their size.

3.3.1. Reformulating the problem

In this section, let $p \in \Lambda^i_{par}$, $q \in \Lambda^j_{par}$. Recall from (3.9) that when $i \neq j$, $\mathcal{SG}_{ij} \simeq \Gamma_p\backslash\Gamma/\Gamma_q$, and $\mathcal{SG}_{ii} \simeq \Gamma_p\backslash(\Gamma - \Gamma_p)/\Gamma_p$. We prove

Lemma 3.3.1. *When $\Re s > \delta(\Gamma, V_0)$, integrating on horospheres in $\widetilde{M}$,*

$$\phi_{ij}(s) = b_j^s \sum_{[\gamma]\in\mathcal{SG}_{ij}} \int_{H(\gamma q, b_j)} P_N(\cdot, p, s)d\mu(\theta) + \mathcal{O}(s^{1/2-N}b_i^s b_j^s). \tag{3.48}$$

The constants are uniform in sets $\{\Re s > \delta(\Gamma, V_0)+\epsilon\}$, $\epsilon > 0$. What is more, this expansion can be differentiated, differentiating the remainder.

Proof. First, for $\tilde{H}_j(b_j) \simeq \Gamma\backslash H(q, b_j)$ the projected horosphere from cusp Z_j at height b_j, integrating in M, we claim

$$\phi_{ij}(s) = -b_j^{2s-d}\delta_{ij} + \int_{x\in\tilde{H}_j(b_j)} b_j^{s-d}E_{i,N}d\theta^d + \mathcal{O}(s^{1/2-N}b_i^s b_j^s), \tag{3.49}$$

where the remainder can be differentiated. Considering zero Fourier modes of E_i in the cylinder $\Gamma_q\backslash\widetilde{M}$ we see that the formula holds if we replace $E_{i,N}$ by E_i, without remainder. Hence, we only have to estimate

$$\left|\partial_s^m\left(b_j^{s-d}\int E_i - E_{i,N}d\theta^d\right)\right|.$$

The surface measure obtained by disintegrating the riemannian volume on $\{y = b_j\}$ is $\mathrm{d}\mu(\theta) = \mathrm{d}\theta^d/b_j^d$. According to the Sobolev trace theorem, the L^2 norm of a restriction to $\tilde{H}_j(b_j)$ — it is an immersed hypersurface — is controlled by the $H^{1/2}$ norm on M. Using this and theorem 3.3, we obtain that the remainder is bounded up to a constant by

$$b_j^{\Re s-d/2}\sup_{k=0,\dots m}\|\partial_s^k(E_i - E_{i,N})\|_{H^{1/2}(M)} = \mathcal{O}(s^{1/2-N}b_i^{\Re s}b_j^{\Re s}).$$

Now, to go from (3.49) to (3.48), we just have to use the description given by theorem 3.3. Indeed, consider a cube $\mathcal{C}_q$ in $H(q, b_j)$ that is a fundamental domain for the action of $\Gamma_q \simeq \mathbb{Z}^d$. Then

$$\begin{aligned}
\int_{x\in \tilde{H}_j(b_j)} b_j^{s-d} E_{i,N} d\theta^d &= \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} \int_{\mathcal{C}_q} b_j^{s-d} P_N(\gamma\cdot, p, s) d\theta^d \\
&= \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} \int_{\gamma\mathcal{C}_q} b_j^{s-d} P_N(\cdot, p, s) d\theta^d \\
&= \sum_{\substack{[\gamma]\in\Gamma_p\backslash\Gamma/\Gamma_q \\ [\gamma]\neq\Gamma_p}} \sum_{\gamma'\in\Gamma_q} \int_{\gamma\gamma'\mathcal{C}_q} b_j^{s-d} P_N(\cdot, p, s) d\theta^d + \delta_{ij} \int_{\mathcal{C}_p} b_i^{s-d} P_N(\cdot, p, s) d\theta^d \\
&= \sum_{\substack{[\gamma]\in\Gamma_p\backslash\Gamma/\Gamma_q \\ [\gamma]\neq\Gamma_p}} \int_{H(\gamma q, b_j)} b_j^{s-d} P_N(\cdot, p, s) d\theta^d + \delta_{ij} \int_{\mathcal{C}_p} b_i^{s-d} P_N(\cdot, p, s) d\theta^d
\end{aligned}$$

It suffices to observe now that

$$\int_{\mathcal{C}_p} b_i^{s-d} P_N(\cdot, p, s) d\theta^d = b_i^{2s-d}.$$

□

We want to give an asymptotic expansion for each term in (3.48). To be able to use stationary phase, the next section is devoted to giving sufficient geometric bounds on the position of $H(\gamma q, b_j)$ with respect to $W^{u0}(p)$.

3.3.2. Preparation lemmas and main asymptotics

Take $p \in \Lambda^i_{par}$, $q \in \Lambda^j_{par}$, $q \neq p$. We will work in $H(q, b_j) \subset \widetilde{M}$. As an embedded Riemannian submanifold, it is isometric to $\mathbb{R}^d$, and the isometry is given by the θ coordinate; we use this to measure distances on $H(q, b_j)$ unless mentioned otherwise. We are considering

$$b_j^{s-d} \int_{H(q,b_j)} P_N(\cdot, p, s) \mathrm{d}\mu(\theta) = \int_{\mathbb{R}^d} e^{-s(G_p - \log b_j)} \tilde{J}_p f_p^N(s, \theta) \mathrm{d}\theta^d \tag{3.50}$$

At all the points where ∇G_p is not orthogonal to the horosphere, this integral is non-stationary as $|s| \to +\infty$. There is only one point in $H(q, b_j)$ where ∇G_p *is* orthogonal to $H(q, b_j)$; it is exactly the point where the geodesic $c_{p,q}$ from p to q intersects $H(q, b_j)$ for the first time — the second is q. It is reasonable to expect that the behaviour of the approximate Poisson kernel around this point will determine the asymptotics of the integral.

It is indeed the case, as we will show that G_p, $\tilde{J}_p$ and f_p^N satisfy appropriate symbol estimates on $H(q, b_j)$. If $a \in C^\infty(\mathbb{R}^d)$, we say that a is a *symbol* of order $n \in \mathbb{Z}$ if for all $k \in \mathbb{N}$,

$$|\langle x\rangle^{-n+k} \partial^k a(x)|_{L^\infty(\mathbb{R}^d)} < \infty \quad \text{where } \langle x\rangle^2 = 1 + x^2. \tag{3.51}$$

For a geodesic coming from p intersecting $H(q, b_j)$, call the first intersection the *point of entry* and the second one the *exit point* — they may be the same. We can assume that the point of entry of $c_{p,q}$ is 0 in the θ coordinate — denoted 0_θ. It is also the point where G_p attains its minimum on $H(q, b_j)$, and this is $\mathcal{T}(c_{p,q}) + \log b_j$, with $\mathcal{T}(c_{p,q})$ the sojourn time as defined in (3.10). We start with a lemma :

Lemma 3.3.2. *The set of entry points $\mathcal{I} \subset H(q, b_j)$ is compact. Its radius is bounded independently from p, q, for the distance on $H(q, b_j)$ given by $H(q, b_j) \simeq \mathbb{R}^d$.*

Proof. First, we prove it is compact. By continuity, $\mathcal{I}$ contains a small neighbourhood U of 0_θ. Let U' be the set of exit points of geodesics whose entry point is in U. It is a neighbourhood of q in $H(q, b_j)$ — by definition of the *visual* topology on $\overline{M}$. The complement of U' has to contain $\mathcal{I}$, and it is compact, so $\mathcal{I}$ is relatively compact. The claim follows because $\mathcal{I}$ is closed.

Now, since G_p is C^∞, and the horosphere is smooth, the boundary of $\mathcal{I}$ only contains points where ∇G_p is tangent to $H(q, b_j)$. Take such a point θ, and consider the triangle with vertices p, 0_θ, and θ. Let α be the angle at 0_θ, and L the distance between 0_θ and θ in $\widetilde{M}$. Since the horoball $B(q, b_j)$ is convex, $\alpha > \pi/2$. From the remark on obtuse triangles (3.15) for $[p, \theta, 0_\theta]$, if $l = G_p(\theta) - G_p(0_\theta)$, we have $L - l = \mathcal{O}(1)$.

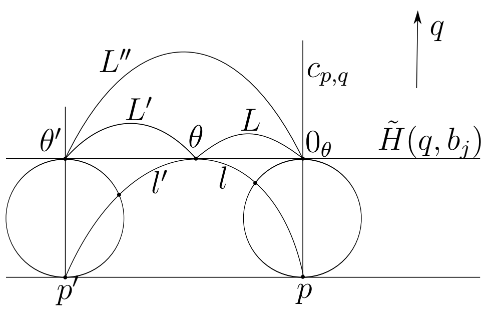

Figure 3.3: the situation.

We want to prove that L is bounded independently from p, q. To see that, consider p' the other endpoint of the geodesic through p and θ, and θ' its projection on $H(q, b_j)$. Let $l' = G_{p'}(\theta) - G_{p'}(\theta')$ and let L' be the distance in $\widetilde{M}$ between θ and θ'. By the same argument, $L' = l' + \mathcal{O}(1)$. Moreover, $l + l'$ is the distance between $H(p, G_p(0_\theta))$ and $H(p', G_{p'}(\theta'))$. So that if L'' is the distance between $0_\theta \in H(p, G_p(0_\theta))$ and $\theta' \in H(p', G_{p'}(\theta'))$, $L'' > l + l'$. However, by the triangle inequality, $L'' \leqslant L + L'$. We deduce that $L'' = L + L' + \mathcal{O}(1)$. By theorem 4.9 and 4.6 of [HIH77], L'' is bounded by constants depending only on the pinching of M, and so is L.

Additionally, from equation (3.13), we deduce:

$$|\theta| = 2b_j \sinh \frac{L}{2}.$$

□

Our second lemma is the following:

Lemma 3.3.3. *In $\mathcal{I}$, G_p is convex. That is, on $\mathcal{I}$, if $\overline{\alpha}$ is the angle ∇G_p makes with the horosphere $H(q, b_j)$, we have $d_\theta^2 G_p \geqslant (\sin \overline{\alpha} + \sin^2 \overline{\alpha} K_{min})/b_j^2$.*

Proof. Let $\theta \in \mathcal{I}$. Take $\mathbf{u} \in \mathbb{R}^d$ with $|u| = 1$ and $\theta' = \theta + \epsilon\mathbf{u}$ for $\epsilon > 0$ small. We apply Topogonov's theorem to the triangle with vertices θ, θ' and $\tilde{p}$, where $\tilde{p}$ is a point that will tend to p. Let $\alpha(\epsilon)$ be the angle at θ. Then by comparison, we have

$$\cosh(K_{min}d(\theta',\tilde{p})) \geqslant \cosh(K_{min}d(\theta,\tilde{p}))\cosh(K_{min}d(\theta,\theta')) - \sinh(K_{min}d(\theta,\tilde{p}))\sinh(K_{min}d(\theta,\theta'))\cos\alpha(\epsilon).$$

As we let $\tilde{p} \to p$,

$$\frac{\cosh(K_{min}d(\theta',\tilde{p}))}{\cosh(K_{min}d(\theta,\tilde{p}))} \text{ and } \frac{\cosh(K_{min}d(\theta',\tilde{p}))}{\sinh(K_{min}d(\theta,\tilde{p}))} \to \exp(K_{min}(G_p(\theta') - G_p(\theta)))$$

and

$$K_{min}(G_p(\theta') - G_p(\theta)) \geqslant \log\left[\cosh(K_{min}d(\theta,\theta')) - \sinh(K_{min}d(\theta,\theta'))\cos\alpha(\epsilon)\right].$$

Now, we let ϵ go to 0. We have $G_p(\theta') - G_p(\theta) = \epsilon\nabla G_p(\theta).\mathbf{u} + \epsilon^2 d_\theta^2 G_p(\theta).\mathbf{u}^{\otimes 2}/2 + o(\epsilon^2)$. Additionally, $\nabla G_p(\theta).\mathbf{u} = -\cos\alpha(0)/b_j$ and $d(\theta,\theta') \sim \epsilon/b_j$ by (3.13), so the RHS becomes

$$\log\left[1 - \frac{K_{min}\epsilon\cos\alpha(\epsilon)}{b_j} + \frac{K_{min}^2\epsilon^2}{2b_j^2} + o(\epsilon^2)\right] = -\frac{\epsilon K_{min}}{b_j}\cos\alpha(\epsilon) + \frac{\epsilon^2 K_{min}^2}{2b_j^2}(1 - \cos\alpha(0)^2 + o(1)).$$

And we deduce

$$d_\theta^2 G_p(\theta).\mathbf{u}^{\otimes 2} \geqslant \frac{K_{min}}{b_j^2}\sin^2\alpha(0) + 2\frac{(\cos\alpha)'(0)}{b_j}.$$

Now, computing in the hyperbolic space, we find that the angle β at which the geodesic between θ and θ' intersects $H(q,b_j)$ satisfies $\beta \sim \epsilon/2b_j$. If $\overline{\alpha}$ is the angle between $\nabla G_p(\theta)$ and $H(q,b_j)$, we find

$$\cos\alpha = \cos\overline{\alpha}\cos\measuredangle(u,\partial_\theta G_p) + \sin\overline{\alpha}\sin\beta \quad \text{and} \quad (\cos\alpha)'(0) = \frac{\sin\overline{\alpha}}{2b_j}.$$

Finally, we can observe that $\alpha(0) \geqslant \overline{\alpha}$ and

$$d_\theta^2 G_p(\theta) \geqslant \frac{\sin\overline{\alpha} + K_{min}\sin^2\overline{\alpha}}{b_j^2}.$$

□

We have to separate the integral (3.50) into two parts, let us explain how we choose them. The stable and the unstable distributions of the flow φ_t are always transverse. Since they are continuous, the angle between them is uniformly bounded by below by some $\alpha > 0$ in any given compact set of M — we say that they are uniformly transverse. Lifting this to $\widetilde{M}$, the angle is uniformly bounded by below on sets that project to compact sets in M. In particular, this is true on the union of the $H(q,b_j)$ for $q \in \Lambda_{par}^j$.

Now, we can also consider the geodesic flow in the hyperbolic space of dimension $d+1$. It has stable and unstable distributions. The cusp Z_j is the quotient of an open set of that space by a group of automorphisms, so that those stable and unstable distributions project down to subbundles E_{hyp}^s, E_{hyp}^u of TS^*Z_i, invariant by the geodesic flow. We call them the $*$-stable and $*$-unstable manifolds of Z_i. The angle between them is constant equal to $\pi/2$, and they are smooth — even analytic.

By definition of the stable and $*$-stable manifolds, if the trajectory of a point $\xi \in S^*Z_i$ stays in Z_i for all times positive, its stable and $*$-stable manifolds coincide. This is the case of $(0, dG_p)$. As a consequence, there is a small neighbourhood V_1 of 0 in the θ plane, whose size can be taken independent from p, q, where the unstable manifold of $(\theta, dG_p(\theta))$ and its $*$-stable manifolds are uniformly transverse.

By the arguments in the proof of lemma 3.3.2, we see that the set of points of $H(q, b_j)$ that are not exit points of geodesics whose entry points are in V_1, is a compact set. Denote it by V_2. Its radius is also bounded independently from p and q. Now, let $\chi \in C_c^\infty(\mathbb{R}^d)$ take value 1 on V_2, and introduce $1 = \chi + (1-\chi)$ in (3.50), to separate it into (I) and (II).

From theorem 7.7.5 (p.220) in Hörmander [Hör03], we deduce

Lemma 3.3.4. *For each p, q, there are coefficients $A_n(p,q)$ so that for every $N \geqslant 1$*

$$(\mathrm{I}) = \left(\frac{\pi}{s}\right)^{d/2} \exp\left\{\int_{c_{p,q}} V_0 - s\mathcal{T}(c_{p,q})\right\}$$
$$\left[\sum_{n \leqslant N-1} \frac{A_n(p,q)}{s^n} + \frac{1}{s^N}\mathcal{O}\left((1 + (\mathcal{T}(c_{p,g}) + \log b_j)^+)^N\right)\right].$$

We have $A_0(p,q) = 1$, and $A_n(p,q) = \mathcal{O}(1 + (\mathcal{T}(c_{p,q}) + \log b_j)^+)^n$. What is more, the A_n do not depend on N for $n \leqslant N-1$.

Proof. From lemma 3.2.3, we already know that the functions under the integral are smooth, uniformly in p, q. From lemma 3.3.3, we know that the phase is non-degenerate at 0. To apply Hörmander's theorem, we need to check that the derivative $|\partial_\theta G_p|$ is uniformly bounded from below in $V_2 - V_1$.

The general observation is that $|\partial_\theta G_p|$ remains bounded from below if ∇G_p stays away from being the outer normal to $H(q, b_j)$.

Start with $\theta \in V_2$ an exit point. Consider c the geodesic along ∇G_p, going out at θ. The closer to the outer normal $\nabla G_p(\theta)$ is, the longer the time c had to spend in the horoball. Since the set of entry points is uniformly compact, this implies that points where ∇G_p is almost vertical — i.e along ∂_y — have to be far from 0. But V_2 is uniformly bounded, so $|\partial_\theta G_p|$ is bounded by below on the exit points in V_2.

For the entry points that are not exit points, we use the uniform convexity from lemma 3.3.3. By that lemma, $\partial_\theta G_p$ is a local diffeomorphism in $\mathcal{I}' = \{\theta,\ \nabla G_q(\theta).\nabla G_p(\theta) < 0\}$. On the boundary of $\mathcal{I}'$, $|\partial_\theta G_p| = b_j^2$. By continuity, there is $\epsilon > 0$ such that $|\partial_\theta G_p(\theta)| < b_j^2/2$ implies $d(\theta, \partial\mathcal{I}') > \epsilon$. As a consequence, from the local inversion theorem, there is $0 < \epsilon' < \epsilon$ and $\epsilon'' > 0$ such that if $|\partial_\theta G_p(\theta)| < b_j^2/2$,

$$B(\partial_\theta G_p(\theta), \epsilon'') \subset \partial_\theta G_p(B(\theta, \epsilon')).$$

Then, when $|\partial_\theta G_p(\theta)| < \epsilon''$, θ has to be at most at distance ϵ' from a zero of $\partial_\theta G_p$, i.e 0_θ.

The constants ϵ' and ϵ'' can be estimated independently from p and q.

Now, we have an expansion

$$(I) = \left(\frac{2\pi}{s}\right)^{d/2} \exp\{-s\mathcal{T}(c_{p,q})\}\left(C_0 + \frac{1}{s}C_1 + \dots\right) \tag{3.52}$$

We have

$$C_0 = \frac{\tilde{J}_p(0_\theta)}{(\det d^2_\theta G_p(0_\theta))^{1/2}}. \tag{3.53}$$

We factor out C_0 from the sum, and define $A_n(p,q) = C_n/C_0$. From lemma 3.2.3 and the fact that ∇F_p is $\mathscr{C}^\infty(\widetilde{M})$, it is quite straightforward to prove the estimates on the A_n's.

Now, we have to compute C_0. From [HIH77, proposition 3.1], we see that

$$\nabla^2 G_p(x) = \mathbb{U}_{x,\nabla G_p(x)}. \tag{3.54}$$

Where $\mathbb{U}$ was introduced after equation (3.33).

We use a simple trick. Along the geodesic $c_{p,q}$, $\nabla(G_p + G_q) = 0$, so that the Hessian $d^2(G_p+G_q)$ is well defined along $c_{p,q}$. This implies that $d^2_\theta(G_p+G_q) = \nabla^2(G_p+G_q)$. But on the horosphere $H(q,b_j)$, G_q is constant, and we find $d^2_\theta G_p(0_\theta) = \nabla^2 G_p(0_\theta) + \nabla^2 G_q(0_\theta)$.

The unstable Jacobi fields along $c_{q,p}$ are the *stable* Jacobi fields along $c_{p,q}$ so $\mathbb{U}_{x,\nabla G_q(x)} = -\mathbb{S}_{x,\nabla G_p(x)}$. Hence,

$$d^2_\theta G_p(0_\theta) = \mathbb{U}_{x,\nabla G_p(x)} - \mathbb{S}_{x,\nabla G_p(x)}. \tag{3.55}$$

In constant curvature, this is the constant matrix $2 \times \mathbf{1}$.

Now, we give another expression for $\tilde{J}^2_p(0_\theta)$. Let $x \in \widetilde{M}$. Consider $\mathbb{J}_u$ the unstable Jacobi field along $(\varphi^p_t(x))$, that equals $(1/y)\mathbf{1}$ for a point along the orbit that is close enough to p — where y is the height coordinate $\exp -G_p$. Then

$$\tilde{J}^2_p(x) = \frac{e^{d.G_p(x)}}{\det \mathbb{J}_u(x)}. \tag{3.56}$$

When $x = 0_\theta$, for $t > 0$, we can write $\mathbb{J}_u(\varphi^p_t(0_\theta)) = Ae^t + Be^{-t}$. We can also define $\mathbb{J}_s$ the stable Jacobi field along $\varphi^p_t(0_\theta)$ that equals $\mathbf{1}$ at 0_θ. From the equation (3.34), we find that

$$W := \mathbb{J}_u(t)^T(\mathbb{U}(t) - \mathbb{S}(t))\mathbb{J}_s(t) \text{ is constant.} \tag{3.57}$$

Hence

$$C_0^2 = \frac{e^{d.G_p(0_\theta)}}{\det W}\det \mathbb{J}_s(0).$$

The limit for $t \to +\infty$ gives $W = 2A$. Whence

$$C_0^2 = \frac{e^{d.G_p(0_\theta)}}{2^d \det A}. \tag{3.58}$$

On the other hand,

$$\begin{aligned}\exp\left\{\int_p^q -2V_0\right\} &= \lim_{t\to+\infty} \det\left(d\varphi_{t|E^u(v)}\right)e^{-td} \text{ for } v \in [p,q] \text{ sufficiently close to } p. \\ &= \det\left\{Ae^{-G_p(0_\theta)}\right\} \text{ from formulae (3.35) and (3.37).}\end{aligned}$$

We conclude that

$$C_0 = 2^{-d/2}\exp\left\{\int_{c_{p,q}} V_0\right\}. \tag{3.59}$$

□

3.3.3. Estimating the remainder terms

Now, we consider

$$(\mathrm{II}) := \int_{\mathbb{R}^d} e^{-s(G_p-\log b_j)}\tilde{J}_p f_p^N(s,\theta)(1-\chi)d\theta = \frac{1}{s^k}\int_{\mathbb{R}^d} e^{-sG_p} L_q^k(\tilde{J}_p f_p^N(s,\theta)(1-\chi))d\theta \quad (3.60)$$

where $L_p f = \operatorname{div}\left[\frac{\partial_\theta G_p}{\|\partial_\theta G_p\|^2} f\right]$. This holds for any $k \in \mathbb{N}$; if we get symbolic estimates on the integrand, we will find that $(\mathrm{II}) = \mathcal{O}(s^{-\infty})(\mathrm{I})$. Our next step is to study the growth of G_p as $\theta \to \infty$.

Lemma 3.3.5. *The function* $\frac{\partial_\theta G_p}{\|\partial_\theta G_p\|}$ *is a symbol of order* 1 *in* θ *in* $\mathbb{R}^d - V_2$*, bounded independently from* p, q*.*

Additionally, $\exp(-s(G_p - \log b_j))$ *is integrable, and for any* $\epsilon > 0$*, there is a constant* $C_\epsilon > 0$ *such that whenever* $\Re s > d/2 + \epsilon$*,*

$$\int_{\mathbb{R}^d - V_2} e^{-s(G_p-\log b_j)}d\theta \leqslant C_\epsilon e^{-\Re s \mathcal{T}(c_{p,q})}.$$

Proof. With each $\theta \in \mathbb{R}^d - V_2$ we associate the point of entry θ_0 — $\theta_0 \in V_1$ by definition. Consider the geodesic coming from p, entering the horoball at θ_0 and going out at θ. Then, if ϵ is the angle of this geodesic with the normal to the horosphere,

$$|\theta - \theta_0| = \frac{2b}{\tan\epsilon}.$$

but, we also have that

$$|\partial_\theta G_p| = \frac{\sin\epsilon}{b}.$$

Hence

$$\frac{1}{|\partial_\theta G_p|} = \frac{1}{2}|\theta-\theta_0|\sqrt{1+\frac{4b^2}{|\theta-\theta_0|^2}},$$

and

$$\frac{\partial_\theta G_p}{\|\partial_\theta G_p\|^2} = \frac{1}{2}(\theta-\theta_0)\sqrt{1+\frac{4b^2}{(\theta-\theta_0)^2}}.$$

It suffices to see that $\theta \mapsto \theta_0$ is a symbol of order -1 to obtain the first part of the lemma. But $\theta \mapsto \theta_0$ is a one-to-one map, and by means of an inversion in the hyperbolic space, we see that $\theta_0 \to \theta/\|\theta\|^2$ is a smooth map. Its derivatives are controlled by the angle that ∇G_p makes with the vertical (and its derivatives). As a consequence $\theta \to \theta_0$ is a symbol of order -1, uniformly in p and q.

Then, using formula (3.13), as $\theta \to \infty$,

$$G_p = 2\log\frac{|\theta|}{2b} + G_p(\theta_0) = 2\log\frac{|\theta|}{2b} + \mathcal{T}(c_{p,q}) + \log b_j + o(1)$$

where the remainder is a symbol of order -1. We deduce that $\exp -sG_p$ is integrable ($\Re s > d/2$), and

$$\int_{\mathbb{R}^d} \mathrm{d}\theta^d e^{-\Re s(G_p-\log b_j)} \leqslant Ce^{-\Re s\mathcal{T}(c_{p,q})}.$$

□

Lemma 3.3.6. *On the horosphere $H(q, b_j)$, $\tilde{J}_p$ is a symbol of order 0 with respect to θ. In symbol norm, it is $\mathcal{O}(\tilde{J}_p(0_\theta))$.*

Proof. We use Jacobi fields and notations introduced in section 3.2.2. We also use the uniform transversality condition in the definition of V_1 — see page 88. In the neighbourhood V_1, since E^u is transverse to the constant curvature stable direction, there exists a smooth matrix $A(\theta)$ such that

$$E^u(\theta) = \{X^+ + X^- | \; X^+ \in E^s_{hyp},\; X^- \in E^u_{hyp},\; X^+ = A(\theta)X^-\}.$$

When we transcribe this to Jacobi field coordinates,

$$E^u(\theta) = \{((\mathbf{1} + A)\xi^\perp, (\mathbf{1} - A)\xi^\perp) | \xi^\perp \in \perp\}$$

Remark here that $(\mathbf{1} + A)$ is invertible ; indeed, if it were not, there would be an unstable Jacobi field on M that would vanish at some point. But a Jacobi field that vanishes at some point cannot go to 0 as $t \to -\infty$, it has to grow.

Now, we consider a trajectory entering the horoball at θ_0. We use the coordinates (θ_0, t) to refer to $\varphi^p_t(\theta_0)$. We use parallel transport to work with vectors in $TS^*M_{|V_1}$. We have

$$E^u(\theta_0, t) = \{X^+ + X^- | \; X^+ \in E^s_{hyp},\; X^- \in E^u_{hyp},\; X^+ = e^{-2t}A(\theta_0)X^-\}$$

and in the horizontal-vertical coordinates

$$E^u(\theta_0, t) = \{((\mathbf{1} + e^{-2t}A)\xi, (\mathbf{1} - e^{-2t}A)\xi) | \xi \perp d/dt\}$$

Actually, for $t \in [0, T]$, the jacobian

$$\text{Jac } \varphi^p_t(\theta_0)$$

is the determinant of $J(0) \mapsto J(t)$ where $J(t)$ are the unstable Jacobi fields along the trajectory $\varphi^p_t(\theta_0)$. From the description with matrix A above, we deduce that this is

$$e^{t.d} \det \frac{\mathbf{1} + e^{-2t}A(\theta_0)}{\mathbf{1} + A(\theta_0)}$$

As a consequence,

$$\frac{\tilde{J}_p(\varphi^p_t(\theta_0))}{\tilde{J}_p(\theta_0)} = \sqrt{\det \frac{\mathbf{1} + e^{-2t}A(\theta_0)}{\mathbf{1} + A(\theta_0)}}.$$

Recall that $t \sim 2 \log|\theta|$ when the trajectory reaches the horosphere again, and that $\theta \to \theta_0$ is a symbol of order -1. We deduce that $\tilde{J}_p(\theta)$ is a symbol of order 0. □

Lemma 3.3.7. *For all $n \geqslant 0$, in the region of the horoball corresponding to trajectories entering in V_1, we can write*

$$f_{n,p}(\theta_0, t) = \tilde{f}_{n,p}(\theta_0, e^{-2t}).$$

We have for all $k \geqslant 0$,

$$\|\tilde{f}_{n,p}\|_{C^k} \leqslant C_{n,k}((\mathcal{T}(c_{p,q}) + \log b_j)^+)^n$$

with $C_{n,k}$ not depending on p nor on q.

Proof. We start by considering two functions a_1 and a_2 of θ_0 and e^{-2t}. Then

$$e^{2t}\nabla a_1.\nabla a_2 \text{ and } e^{2t}\Delta a_1$$

are still smooth functions of θ_0 and e^{-2t}. Consider a trajectory $x(t) = (\theta_0, t)$. We can take normal coordinates along this geodesic (t, x'). We then only need to prove that $e^t\partial\theta_0/\partial x'_{|x'=0}$ is a smooth function of θ_0 and t. First, we observe that $\partial\theta_0/\partial x'_{|x'=0,t=0}$ is only controlled by the angle between the geodesic and the horosphere $H(q, b_j)$, and this angle we have shown to be smooth. We only have to consider $\partial x'(t)/\partial x'(0)_{|x'=0}$, that is, the differential of the flow φ_t^p transversally to ∇G_p. We have computed it in the previous proof; it is $e^t(\mathbf{1} + e^{-2t}A(\theta_0))(\mathbf{1} + A(\theta_0))^{-1}$.

Now, we proceed by induction on n. First, $f_{0,p} = 1$ so it obviously satisfies the assumptions; it is also the case of $F_p = \log \tilde{J}_p$. Assume that the hypothesis has been verified for some $n \geqslant 0$. Then by the above and (3.41), 3.2.3, $e^{2t}Q_p f_{n,p}$ is a smooth function of θ_0 and e^{-2t}, with the same control as for $f_{n,p}$, and

$$f_{n+1,p}(\theta_0, t) = f_{n+1,p}(\theta_0, 0) + \frac{1}{2}\int_0^t Q_p f_{n,p} \mathrm{d}s.$$

we can write the integral as

$$\int_0^t e^{-2s} a(\theta_0, e^{-2s})\mathrm{d}s = \left[\int a(\theta_0, \rho)\mathrm{d}\rho\right]_{e^{-2t}}^1,$$

for some smooth function a. This ends the proof. □

Now, recall that $e^{-2t} \sim |\theta - \theta_0|^{-4}$, so this proves that $f_{n,p}(\theta)$ is a symbol of order 0 as $\theta \to \infty$.

Putting lemmas 3.3.5, 3.3.6, 3.3.7 together, we deduce from equation (3.60) that for all $N, k > 0$ and $\epsilon > 0$, there is a constant $C_{N,k,\epsilon} > 0$ such that, when $\Re s > d/2 + \epsilon$,

$$|(\mathrm{II})| \leqslant C_{N,k} e^{-\mathcal{T}(c_{p,q})\Re s}(1 + (\mathcal{T}(c_{p,q}) + \log b_j)^+)^N s^{-k}. \tag{3.61}$$

3.3.4. Main result

With the notations of lemma 3.3.4, for $c \in \pi_1^{ij}(M)$ with endpoints p, q in $\widetilde{M}$, we define

$$a^n(c) := \exp\left\{\int_c V_0\right\} A_n(p, q). \tag{3.62}$$

We also define

$$\mathcal{T}_{ij}^0 := \inf\{\mathcal{T}(c),\ c \in \pi_1^{ij}(M)\} \text{ and } \mathcal{T}_{ij}^\# = \min(-\log b_i b_j, \mathcal{T}_{ij}^0). \tag{3.63}$$

Putting together lemmas 3.3.1, 3.3.4, and equation (3.61), we get

Theorem 3.4. *For two cusps Z_i and Z_j not necessarily different, and for every $N > 0$, when $\Re s > \delta(\Gamma, V_0)$,*

$$\phi_{ij}(s) = \left(\frac{\pi}{s}\right)^{d/2} \sum_{[c]\in\pi_1^{ij}(M)} \sum_{n=0}^{N-1} \frac{1}{s^n}\frac{a^n(c)}{e^{s\mathcal{T}(c)}} + \frac{\mathcal{O}(1)}{s^N e^{s\mathcal{T}_{ij}^\#}}. \tag{3.64}$$

We proceed to give a parametrix for φ. When taking the determinant of the scattering matrix $\phi(s)$, we use the Leibniz formula

$$\varphi(s) = \sum_{\sigma} \varepsilon(\sigma) \prod_{i=1}^{\kappa} \phi_{i,\sigma(i)}(s).$$

The sum is over the permutations σ of $[\![1, \kappa]\!]$, and $\varepsilon(\sigma)$ is the signature of σ. The remainder will be bounded by terms of the form

$$s^{\kappa/2-N} \exp\left\{-s\left(\mathcal{T}^{\#}_{1\sigma(1)} + \cdots + \mathcal{T}^{\#}_{\kappa\sigma(\kappa)}\right)\right\}.$$

This one corresponds to the error of approximation for the product $\phi_{1\sigma(1)}(s) \dots \phi_{\kappa\sigma(\kappa)}(s)$ where σ is a permutation of $[\![1, \kappa]\!]$. Hence, we define

$$\mathcal{T}^{\#} = \min_{\sigma} \sum \mathcal{T}^{\#}_{i\sigma(i)}. \tag{3.65}$$

It corresponds to the slowest decreasing remainder term as $\Re s \to +\infty$. Recall the definition in (3.11): the scattering cycles ($\mathcal{SC}$) are numbers of the form $T_1 + \cdots + T_\kappa$, where $T_i \in \mathcal{ST}_{i\sigma(i)}$. We define $\mathcal{T}^0$ to be the smallest scattering cycle. It corresponds to the slowest decreasing term in the parametrix. By definition, $\mathcal{T}^{\#} \leqslant \mathcal{T}^0$.

Remark 3.7. *There are two cases. When $\mathcal{T}^{\#} < \mathcal{T}^0$, the error is bigger than the main term in the parametrix, for $\Re s$ too big with respect to $\Im s$. That occurs when the incoming plane waves from the cusps encounter variable curvature before they have travelled the shortest scattered geodesics.*

When $\mathcal{T}^{\#} = \mathcal{T}^0$, the error term is always smaller than the main term in the parametrix. This means that the variations of the curvature happen not too close to the cusps. It is in particular the case when the curvature is constant.

In any case, let $\lambda_{\#} = \exp \mathcal{T}^{\#}$. Also let $\lambda_0 < \lambda_1 < \cdots < \lambda_k < \dots$ be the ordered elements of $\{\exp \mathcal{T},\ \mathcal{T} \in \mathcal{SC}\}$. We can now state the conclusion of this section:

Theorem 3.5. *There are real coefficients $\{a_k^n\}_{k,n\geqslant 0}$ such that if*

$$L_n := \sum_{k\geqslant 0} \frac{a_k^n}{\lambda_k^s},$$

all the L_n's converge in the half plane $\{\Re s > \delta(\Gamma, V_0)\}$. In that half plane, for all $N \geqslant 0$,

$$\varphi(s) = s^{-\kappa d/2} \left[\sum_{n=0}^{N} s^{-n} L_n(s) + \frac{\mathcal{O}(1)}{s^{N+1}\lambda_{\#}^s} \right].$$

3.4. Dependence of the parametrix on the metric

This section is devoted to studying the regularity of the coefficients $a^n(c)$ with respect to the metric. We prove that they are continuous in the appropriate spaces in sections 3.4.1 and 3.4.2. Then, we prove an openness property in $\mathscr{C}^\infty$ topology on metrics. While essential to the proof of theorem 3.1, this part is quite technical, and the impatient reader may skip directly to section 3.5.

3.4.1. The marked *Sojourn Spectrum*

As announced in section 3.1.2, we emphasize the dependence of objects on the metric from now on. In particular, when we write $[\overline{c}_g] \in \pi_1^{ij}(M)$, we mean that we take some class in $\pi_1^{ij}(M)$, and consider $\overline{c}_g$, the unique scattered geodesic for g in that class.

We denote by T_g the application $T_g : [\overline{c}_g] \in \pi_1^{ij}(M) \mapsto \mathcal{T}(\overline{c}_g)$. We also write $a^n(g, [c])$ instead of just $a^n(c)$.

In what follows, we are interested in the regularity and openness properties of φ. It is obtained as the determinant of $\phi(s)$. Since the determinant is a polynomial expression, it is certainly smooth and open with respect to ϕ. As a consequence, it suffices to study the regularity of each ϕ_{ij} independently, and the openess properties of $\phi(s)$ instead of φ.

The following lemma is classical:

Lemma 3.4.1. *Let $(g_\epsilon)_{\epsilon\in\mathbb{R}}$ be a family of $\mathscr{C}^\infty$ cusp metrics on M, so that their curvature varies in a compact set independent from ϵ. Suppose additionally that g_ϵ is C^{2+k} on $\mathbb{R}\times M$ for $k \geqslant 0$. Then we say that g_ϵ is a C^{2+k} family of metrics.*

In that case, the geodesic flow $\varphi_t^{g_\epsilon}$ is C^{1+k} on $\mathbb{R} \times M$ for $k \geqslant 0$.

This is the direct consequence of

Lemma 3.4.2. *Let f be a C^k function on $\mathbb{R} \times U \subset \mathbb{R}^m$ where $k \geqslant 1$ and U is an open set. Then the flow associated to*

$$\dot{x} = f(t, x).$$

is C^k.

The proof of this can be found in any introduction to dynamical systems. Now, we can prove that both the marked set of scattered geodesics and the marked Sojourn Spectrum are continuous along a perturbation of the metric that is at least C^1 in the C^2 topology on metrics.

Lemma 3.4.3. *Let g_ϵ be a C^{2+k} $(k \geqslant 0)$ family of cusp metrics on M. Let $\overline{c}$ be a scattered geodesic for $g = g_0$. Then there is a C^{1+k} family of curves $\overline{c}_\epsilon$ on M such that $\overline{c}_\epsilon$ is a scattered geodesic for g_ϵ. In particular, this proves that $g \mapsto \overline{c}_g$ (given a class in $\pi_1^{ij}(M)$) $g \mapsto T_g$ are C^{1+k} in C^{2+k} topology on g, when $k \geqslant 0$.*

Proof. Let us assume that $\overline{c}$ enters M in Z_i and escapes in Z_j. We can assume that the variations of the curvature of g_ϵ always take place below $y = y_0$. Let x_0 (resp. x_1) be the point where $\overline{c}$ intersects the projected horosphere H_i (resp. H_j) at height y_0 in Z_i (resp. Z_j), entering (resp. leaving) the compact part. For $x \in H_i$ and ϵ close to 0, we can consider the following curve: $c_{x,\epsilon}$ is the geodesic for g_ϵ, that passes through x, and is directed by $-\partial_y$ at x. We have $\overline{c} = c_{x_0,0}$. For (x, ϵ) close enough to $(x_0, 0)$, $c_{x,\epsilon}$ intersects the projected horosphere H_j, for a time close to $\mathcal{T}(\gamma) + 2\log y_0$. We let $x'(x, \epsilon)$ be that point of intersection, and $v(x, \epsilon)$ the vector $c'_{x,\epsilon}$ at $x'(x, \epsilon)$.

Now, by the lemma above, $v(x, \epsilon)$ is C^{1+k}, and by the Local Inversion Theorem, there is a unique $\epsilon \mapsto x(\epsilon)$, C^{1+k}, such that $v(x(\epsilon), \epsilon)$ is the vertical for all ϵ sufficiently close to 0, as soon as $\partial_x v(0, 0)$ is invertible. But the fact that it is invertible is a direct consequence of the non-degeneracy of the phase function shown in lemma 3.3.3. □

Let

$$L_n^{ij}(s) = \sum_{[\bar{c}]\in\pi_1^{ij}(M)} \frac{a^n(g,[\bar{c}])}{e^{sT_g([\bar{c}])}}$$

Lemma 3.4.4. *Let g_ϵ be a C^{2+k} family of metrics, $k \geqslant 0$. Then, as a formal series, L_0^{ij} depends on ϵ in a C^k fashion. In particular, the series L_0 giving the first asymptotics for φ at high frequencies, also depends in a C^k fashion on ϵ.*

Proof. We only have to prove that $a^0(g_\epsilon, [\bar{c}])$ depends on ϵ in a C^k fashion. Since V_0 is only a Hölder function, it is easier to study the regularity of a^0 with the original expression (3.53). That is, we have to study $d_\theta^2 G_p(0_\theta)$ and $\tilde{J}_p(0_\theta)$.

First, consider $\tilde{J}_p$. It is a function of the jacobian of the flow φ_t^p along $\bar{c}$. Since φ_t^p is just some restriction of φ_t, φ_t^p is C^{1+k} on $\mathbb{R}\times M$. We also have that $\bar{c}$ is C^{1+k}, so that $\tilde{J}_p$ is C^k on ϵ.

For $d_\theta^2 G_p(0_\theta)$, consider that it is obtained as the first variation of ∇G_p along the horocycle $H(q, b_j)$. But this means that $d_\theta^2 G_p(0_\theta)$ is again obtained directly in terms of $d\varphi_t^p$ along $\bar{c}$ and this ends the proof. □

Here already, we see that the first order behaviour of the scattering determinant at high frequency ($\Re s$ bounded and $\Im s \to \pm\infty$) depends continuously on g in C^2 topology. The next section is devoted to studying this regularity for other terms.

3.4.2. Higher order coefficients of the parametrix

Now, we are interested in the regularity of $a^n(g, [c])$ for $n \geqslant 1$.

Lemma 3.4.5. *Let g_ϵ be a C^{2+k} family of cusp metrics on M. Then the coefficients $a^n(g,[c])$ depend in a C^{k-2n} fashion on ϵ, as soon as $k \geqslant 2n$.*

In the following proof, we fix two points p, q on the boundary. Most of the functions that appear depend on p and q, but to simplify notations, we omit that dependence. We do not fix n, but k will always be assumed to be greater or equal to $2n$.

Proof. Let us start by a discussion of classical stationary phase in $\mathbb{R}^d$. Let σ be smooth compactly supported function on $\mathbb{R}^d$. Then, as $|s| \to +\infty$ with $\Re s > 0$,

$$\int_{\mathbb{R}^d} e^{-sx^2}\sigma(x)dx = \pi^{d/2}s^{-d/2}\left[\sigma(0) + \frac{1}{4s}\Delta\sigma(0) + \cdots + \frac{1}{n!4^n s^n}\Delta^n\sigma(0) + \mathcal{O}(s^{-n-1})\right]$$

where $\Delta^n\sigma = \Delta\ldots\Delta\sigma$. If G is a non-degenerate phase function around 0, we find Ψ smooth around 0 such that $G\circ\Psi(x) = x^2$, by Morse theory. Then, if σ is still compactly supported but has an expansion $\sigma(s,x) \sim \sigma_0(x) + \sigma_1(x)/s + \cdots + \sigma_n(x)/s^k + \ldots$, we find

$$\int_{\mathbb{R}^d} e^{-sG(x)}\sigma(s,x)dx \sim \left(\frac{\pi}{s}\right)^{d/2}\left[\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} s^{-n-l}\frac{1}{l!4^l}\Delta^l(\sigma_n\circ\Psi.\operatorname{Jac}(\Psi))(0)\right].$$

In other words, the coefficient of $\pi^{d/2}s^{-d/2-n}$ is

$$\sum_{l=0}^{n}\frac{1}{l!4^l}\Delta^l(\sigma_{n-l}\circ\Psi.\operatorname{Jac}(\Psi)) \tag{3.66}$$

It is a well known fact that the Morse chart Ψ is not uniquely defined. However, from the computations above, the operators

$$\sigma \mapsto \Delta^j(\sigma \circ \Psi.\, \mathrm{Jac}\, \Psi)(0)$$

do not depend on the choice of Ψ, but only on G. By writing the condition $G \circ \Psi = x^2$, one can see that $d\Psi^T(0).d\Psi(0) = d^2G(0)$. This determines $d\Psi(0)$. The higher order derivatives of Ψ are undetermined, but one can see that they can be chosen recursively, so that $d^k\Psi(0)$ only depends on the $(k+1)$-jet of G at 0.

We apply the discussion above to $a^n(g,[c])$. The notations are coherent with section 3.3.2 and equation (3.50) if one set $G = G_p$ and $\sigma = \tilde{J}_p f_p^N$. We decompose a^n, following equation (3.66). We find that a^n depends on derivatives of $\tilde{J}$, f and G. We expand each summand in the decomposition, using the Leibniz rule. Then we gather the terms involving the highest order derivatives of the metric. They are

$$\frac{1}{n!4^n}\Big(\mathrm{Jac}\,\Psi(0)\Delta_\theta^n \tilde{J}(0) + \tilde{J}(0)\Delta_\theta^n \,\mathrm{Jac}\,\Psi(0)\Big) + \tilde{J}(0)\,\mathrm{Jac}\,\Psi(0)\sum_{l=1}^{n}\frac{1}{(n-l)!4^{n-l}}\Delta_\theta^{n-l}(f_l \circ \Psi). \tag{3.67}$$

In the proof of 3.4.4, we saw that along a C^{2+k} perturbation, $\tilde{J}$ is C^k, so that $\Delta_\theta^n(\tilde{J}\circ\Psi)$ is C^{k-2n}. By the same argument, we find that the $(2n+2)$-jet of G at 0_θ is a C^{k-2n} function of ϵ, so that $\Delta_\theta^n \,\mathrm{Jac}\,\Psi$ is also C^{k-2n}.

Remark. *One can check that the numbers $\Delta^j \,\mathrm{Jac}\,\Psi(0)$ are, up to universal constants, the Taylor coefficients in the expansion of the function* $\mathrm{vol}(G \leqslant r^2)$.

Now, we deal with the f_n's. From the definition of Q in (3.41) and f_n in (3.42), we can prove by induction that for $n \geqslant 1$,

$$f_n = \frac{1}{2^n}\int_{-\infty}^{0} dt_n \left[\int_{t_n}^{0} dt_{n-1}\cdots\int_{t_2}^{0} dt_1 Q_{t_{n-1}}\cdots Q_{t_1}Q_0 f_0\right]\circ\varphi_{t_n}^p \tag{3.68}$$

where Q_t is defined by $Q(f\circ\varphi_t^p) = (Q_t f)\circ\varphi_t^p$. Since F is essentially a jacobian of φ_t^p, it is C^k on $\mathbb{R}\times M$ along a C^{2+k} perturbation. From the formula (3.68), we deduce that f_n is C^{k-2n} along a C^{2+k} perturbation when $k \geqslant 2n$, and this ends the proof.

□

3.4.3. Openness in C^∞ topology

To find that the coefficients of the parametrix are open, we are going to adopt a different point of view from the previous section. We let $a^{-1}(g,[c]) = T_g([c])$. We aim to prove the following:

Lemma 3.4.6. *Let $[c_1],\dots,[c_N]$ be distinct elements of $\pi_1^{i_1j_1}(M),\dots,\pi_1^{i_Nj_N}(M)$, and take indices $n_1,\dots,n_N$. Then the application*

$$a_N : g \mapsto \big((a^{-1},a^0,\dots,a^{n_1})(g,[c_1]),\dots,(a^{-1},a^0,\dots,a^{n_N})(g,[c_N])\big) \in \mathbb{R}^{\sum n_i+1}$$

is open in C^∞ topology on g.

Proof. First, observe that it suffices to prove that the differential of a_N is surjective. Indeed, we can then use the local inversion theorem to prove the openness property.

For each class $[c_i]$, we will compute the variation of $(a^0, \ldots, a^{n_i})$ along a well chosen smooth family of cusp metrics g_ϵ. We will find that this variation is a linear form in a jet of the variation $\partial_\epsilon g_\epsilon$ along $\overline{c}_i$. From the properties of this linear form, we will find that there are functions with arbitrary compact support on which it does not vanish. This will prove the lemma for $N = 1$.

For the case when $N \geqslant 1$, observe that since the $[c_i]$ are distinct, the $\overline{c}_i$ are also distinct. Then, it suffices to observe that we can take a finite number of small open sets U_i such that $U_i \cap U_j = \varnothing$ when $i \neq j$, and $U_i \cap \overline{c}_i \neq \varnothing$. Then we can perturb in each open set independently, and in this way, we see that the differential of a_N is surjective, and this ends the proof.

Remark 3.8. *There might seem to be a difficulty when the geodesic $\overline{c}$ has a self intersection, because at the point of intersection, we have less liberty on the perturbations we can make. However, we will always choose to perturb away from those intersection points.*

As we have reduced the proof to the case $N = 1$, let $[c] \in \pi_1^{ij}(M)$.

First case, n = 0.

Lemma 3.4.7. *Let g_ϵ be a C^∞ family of cusp metrics. Then*

$$\partial_\epsilon T([c]) = \frac{1}{2}\int (\partial_\epsilon g)_0(\overline{c}_0'(t), \overline{c}_0'(t))dt.$$

In particular, if U is an open set that intersects $\overline{c}_0$, one can find a perturbation of the metric, supported in U, along which $\partial_\epsilon T([c]) \neq 0$.

Proof. From the arguments above, we can construct a variation c_ϵ of $\overline{c}_0$ such that each γ_ϵ is an *unparametrized* geodesic for g_ϵ. We can assume that for t negative (resp. positive) enough, $y_i(c_\epsilon) = y_i(\overline{c}_0)$ (resp. $y_j(c_\epsilon) = y_j(\overline{c}_0)$). Then in local coordinates

$$\partial_\epsilon T([c]) = \frac{1}{2}\int (\partial_\epsilon g)_0(\overline{c}_0'(t), \overline{c}_0'(t)) + 2g_0(\overline{c}_0'(t), \partial_\epsilon c_0'(t))dt$$

In the RHS, the second term, we can interpret as the 1st order variation of the length of the curve c_ϵ for g_0. Since $\overline{c}_0$ is a geodesic, this has to be zero □

Lemma 3.4.8. *The logarithmic differential*

$$d_g \log a^0(g, [c])$$

is non-degenerate on the set of symmetric 2-tensors h on M such that h and dh vanish at $\overline{c}$. This proves the property for $n = 0$, because if h is such a 2-tensor, along the perturbation $g + \epsilon h$, the curve $\overline{c}$ is always a scattered geodesic of constant sojourn time, and $d_g a^{-1}(g, [c]).h = 0$.

Proof. since the curve $\overline{c}$ does not depend on the metric in our context, it is reasonable to use the method of variation of parameters. Let $g_\epsilon = g + \epsilon h$, and consider $\mathbb{J}_{u,\epsilon} =$

$\mathbb{J}_u + \epsilon\tilde{\mathbb{J}}_u + o(\epsilon)$ the unstable Jacobi field for g_ϵ along $\bar{c}$ defined in page 89. We also use $\mathbb{J}_s$ as defined in the same page. We can write

$$\tilde{\mathbb{J}}_u = \mathbb{J}_u\widetilde{A}(t) + \mathbb{J}_s\widetilde{B}(t)$$

and we find the equations for $\widetilde{A}$ and $\tilde{B}$:

$$\begin{aligned}\mathbb{J}_u\widetilde{A}' + \mathbb{J}_s\widetilde{B}' &= 0\\ \mathbb{J}_u'\widetilde{A}' + \mathbb{J}_s'\widetilde{B}' &= -(d_gK.h)\mathbb{J}_u.\end{aligned}$$

Recall from the arguments in page 89 that

$$a^0(g_\epsilon,[\bar{c}])^2 = \lim_{t\to+\infty}\frac{e^{d.G_p(0_\theta)}}{\det \mathbb{J}_u\mathbb{J}_s} = a^0(g_\epsilon,[\bar{c}])^2\frac{1}{\det \mathbf{1}+\epsilon\widetilde{A}(+\infty)}.$$

where $\widetilde{A}(+\infty)$ is the limit of $\widetilde{A}$ when $t\to\infty$. Hence

$$\frac{d}{d\epsilon}\log a^0(g_\epsilon,[\bar{c}]) = -\frac{1}{2}\operatorname{Tr}\widetilde{A}(+\infty).$$

We find

$$\widetilde{A}(+\infty) = \int_{\mathbb{R}}\mathbb{J}_u^{-1}(\mathbb{S}-\mathbb{U})^{-1}(d_gK(t).h)\mathbb{J}_u(t)dt$$

and conclude

$$\frac{d}{d\epsilon}\log a^0(g_\epsilon,[\bar{c}]) = \frac{1}{2}\int_{\mathbb{R}}\operatorname{Tr}\left\{(\mathbb{U}-\mathbb{S})^{-1}(d_gK(t).h)\right\}dt \tag{3.69}$$

When the curvature of g is constant along $\bar{c}$, one may observe that this gives a particularly simple expression. Now we prove that the differential $h\mapsto d_gK.h$ is surjective on the set of symmetric matrices along the geodesic $\bar{c}$. We consider Fermi coordinates along $\bar{c}$. That is, the coordinate chart given by

$$(x_1;x')\mapsto \exp_{\bar{c}(x_1)}\{x'\}\in N.$$

Remark 3.9. *When $\bar{c}$ has self-intersection, this chart is not injective. However, we can assume that h vanishes around such points of intersection, and the computations below remain valid.*

In those coordinates, $g-\mathbf{1}$ and dg vanish along the geodesic, which is $\bar{c}\simeq\{x'=0\}$. We deduce that the Christoffel coefficients Γ_{ij}^k also vanish to second order on $\bar{c}$. Now we recall from [Pau14] two useful formulae.

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l g^{lk}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}) \tag{p.210}$$

$$R(\partial_i,\partial_j)\partial_k = \sum_l\left\{\partial_i\Gamma_{jk}^l - \partial_j\Gamma_{ik}^l + \sum_m \Gamma_{jk}^m\Gamma_{im}^l - \Gamma_{ik}^m\Gamma_{jm}^l\right\}\partial_l. \tag{p.211}$$

Whence we deduce that on $\bar{c}$

$$R_g(\partial_i, \partial_1)\partial_1 = \sum_{l=1}^{d+1} \partial_i \Gamma_{11}^l \partial_l, \\ = -\frac{1}{2}\partial_i\partial_k g_{11}\partial_k. \tag{3.70}$$

We see that $2K(t) = -d^2 g_{11}(t)$, so that $d_g K.h = -1/2d^2 h_{11}$, and this is certainly surjective onto the set of smooth functions along the geodesic valued in symmetric matrices. In particular, the RHS of (3.69) defines a non-degenerate linear functional on the set of compactly supported 2-symmetric tensors along $\bar{c}$. □

This ends the case $n = 0$.

General case, $\mathbf{n \geqslant 1}$. We introduce a special coordinate chart on $\widetilde{M}$:

$$\varsigma_g : (x, t) \in H(p, b_i) \times \mathbb{R} \mapsto \varphi_t^q(x).$$

Since $H(p, b_i) \simeq \mathbb{R}^d$, we are now working in $\mathbb{R}^{d+1}$. In the coordinates ς_g, the flow has a very simple expression: $\varphi_t^p(x, s) = (x, s + t)$. The metric also:

$$\varsigma^* g = \tilde{g}(x, t; dx) + dt^2; \tag{3.71}$$

the jacobian

$$Jac(\varphi_t^q)(x, s) = \sqrt{\frac{\det \tilde{g}(x, s+t)}{\det \tilde{g}(x, s)}}, \tag{3.72}$$

and

$$F(x, s) = \frac{1}{4}\left(\log\det \tilde{g}(x, s) - \log\det \tilde{g}(x, 0) + 2sd\right). \tag{3.73}$$

We can find that $\tilde{g}(x, 0)$ actually does not depend on x. We also have that $G_p(x, s) = s - \log b_i$.

Idea of proof. If we perturb $\tilde{g}$ by a symmetric 2-tensor h on the slices, we obtain a new metric $\tilde{g}_h$ on $\mathbb{R}^{d+1}$. We can obtain a metric g_h^1 on $\widetilde{M}$, pushing forward by ς_g. If the support Ω of this perturbation is small enough that $\gamma\Omega \cap \Omega = \varnothing$ for all $\gamma \neq 1$, we can periodize the perturbation to obtain a metric on M, or equivalently, a metric g_h invariant by Γ on $\widetilde{M}$.

The metric g_h, seen in the chart ς_g, does not have the nice decomposition (3.71) anymore. However, that decomposition still holds in the complement of $\Upsilon := \bigcup_{\gamma \notin \Gamma_p} \gamma\Omega$. If Ω was well chosen, this includes a neighbourhood Ω' of the geodesic $\bar{c}$ that we wanted to perturb.

The condition for Ω to be appropriate is that the projection $\widetilde{M} \to M$ is injective on Ω, and that $\gamma\Omega$ does intersect the lift $[p, q]$ of $\bar{c}$. For this, it suffices that Ω is not too close to the points I in $[p, q]$ that project to self-intersection points of $\bar{c}$. See figure 3.4.

There is a last difficulty. The point 0_θ is represented by $(0, t_1)$ with $t_1 = T_g(g, [c]) + \log b_i b_j$ — see the paragraph after equation (3.51). It is possible that $t_1 < 0$. In that case, the geodesic $\bar{c}$ only encounters constant curvature. To perturb the coefficients, we will need to create variable curvature along the geodesic. In particular, that will change the values of b_i and b_j.

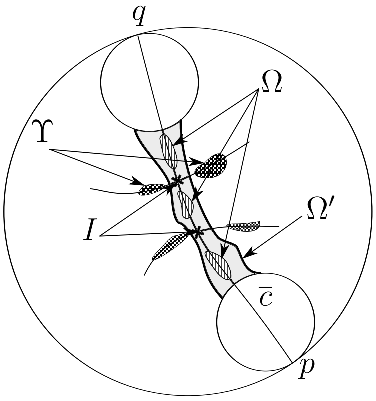

Figure 3.4: Global situation.

To overcome this difficulty, we proceed in the following way. Instead of integrating along the projected horosphere at height b_j in the cusp Z_j, we integrate on the projected horosphere at height $b_j^\star \geqslant b_j$ in the proof of theorem 3.4. We do it so that for all $[\overline{c}] \in \pi_1^{ij}(M)$, $\mathcal{T}(\overline{c}) + \log b_i^\star b_j^\star > 0$ (for all $i, j \dots$). Since the marked sojourn time function is proper, only a finite number of scattered geodesics intervene here. All quantities that depended on b_i, b_j before will now receive a $\star$ when we replace b_i by $b_i^\star$.

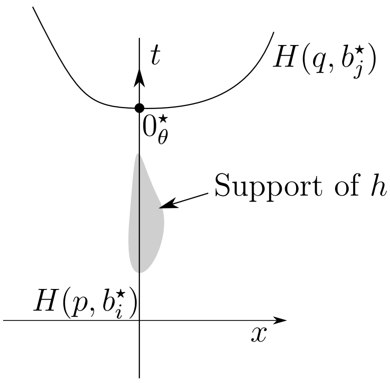

Figure 3.5: A close up.

Coming back to perturbing coefficients, the point $0_\theta^\star$ is represented by $(0, t_1^\star)$ with $t_1^\star = T_g(g, [c]) + \log b_i^\star b_j^\star$. If the perturbation h is compactly supported in $\{0 < t < t_1^\star\}$, the expression of g_h, $H(q, b_j^\star)$, $\Delta_\theta^\star$, $\tilde{J}$, G_p will not depend on h in the chart $\varsigma_g^\star$, around $0_\theta^\star$. In particular, a^{-1} and a^0 are always constant along such a perturbation.

Now, we assume that the change in the slices is ϵh where h is a 2-symmetric tensor such that $h, dh, \dots, d^{2n-1}h$ vanish along $\{x = 0\}$ in the chart $\varsigma_g^\star$, and h is supported for $0 < T < s < T' < t_1^\star$. Let $g_\epsilon := g_{\epsilon h}$.

Lemma 3.4.9. *Under such a perturbation, $a^{-1}, a^0, \dots, a^{n-1}$ are constant.*

Proof. As we saw in section 3.4.2, the coefficient a^k is computed from the $2k - 2\ell$ jet of f_ℓ, $\ell = 1, \dots, k$, at 0_θ, and also the $2k$ jet of $\tilde{J}$ and G. Those computations are done with Δ_θ, which in our chart ς_g has a complicated expression. However, since we are not perturbing the metric around 0_θ, the coefficients of Δ_θ do not change under the perturbation. From equation (3.72), and the expression for G in this chart, we see that the contribution of $\tilde{J}$ and G to a^k will not change under perturbation (independently from the order of cancellation of h).

We are left to prove that the $2k-2\ell$ jet of f_ℓ at 0_θ does not change for $0 \leqslant \ell \leqslant k \leqslant n-1$. From formula (3.73), we see that the $2n-1$ jet of F along $\bar{c}$ will not change along the perturbation. From equation (3.68), we see that the m jet of f_ℓ at 0_θ depends on the $m+2\ell$ jet of F, and the $m+2\ell-1$ jet of g — recall that the coefficients in the Laplacian Δ depend on dg, and the coefficients in ∇ depend on g. Taking this for $m = 2k-2\ell$ and $\ell \leqslant k \leqslant n-1$, we find that the $2k-2\ell$ jet of f_ℓ can be computed with only the $2n-2$ jet of g at $\bar{c}$, and this proves the lemma. □

From the proof of the lemma above, we see that in a^n, the only change will come from the change in the derivatives of order $2n-2k$ of f_k, and more precisely, the parts of these variations that come from the change in $2n$ derivatives of F, in the x direction. As a consequence, we can do all the forecoming computations as if the differential operators appearing had constant coefficients, and replace Δ (resp. $\Delta_\theta(\cdot \circ \Psi) \circ \Psi^{-1}$) by

$$\tilde{\Delta} := \tilde{g}_{ij}(s)\partial_i\partial_j \quad (\text{resp. } \tilde{\Delta}_\theta := c_{ij}\partial_i\partial_j)$$

where the matrices $(\tilde{g}_{ij})(t)$ and (c_{ij}) are symmetric, positive matrices. Recall the metric g has the expression

$$g_{(x,t)}(dx, dt) = \tilde{g}_{x,t}(dx) + dt^2$$

and $(\tilde{g}_{ij})(t)$ is the value of $\tilde{g}_{0,t}^{-1}$, but this fact will not be used later. Recall that the operator Q_t was defined by $(Q_t f) \circ \varphi_t^q = Q(f \circ \varphi_t^q)$. We define $\tilde{\Delta}_t$ in the same way. An easy computation shows that $\tilde{\Delta}_t = \sum g_{ij}(s-t)\partial_i\partial_j$.

Now, we use formula (3.67). We only keep the terms that vary under the perturbation g_ϵ. This yields

$$a^n(g_\epsilon, [\bar{c}]) - a^n(g, [\bar{c}]) = a^0(g, [\bar{c}]) \sum_{l=1}^{n} \frac{1}{(n-l)!4^{n-l}} \tilde{\Delta}_\theta^{n-l} \left\{ (f_l)_\epsilon - f_l \right\}.$$

Next we use equation (3.68), leaving out the constant terms again. We find:

$$\frac{a^n(g_\epsilon, [\bar{c}]) - a^n(g, [\bar{c}])}{a^0(g, [\bar{c}])} = \sum_{l=1}^{n} \frac{1}{(n-l)!4^n 2^{-l}} \int_{\mathfrak{S}} dt_l \ldots dt_1 \, \tilde{\Delta}_\theta^{n-l} \tilde{\Delta}_{t_{l-1}} \ldots \tilde{\Delta}_{t_1} \tilde{\Delta} \left\{ F_\epsilon - F \right\} (0, t_1^\star + t_l)$$

Here, $\mathfrak{S}$ is the simplex $\{-\infty < t_l \leqslant t_{l-1} \leqslant \cdots \leqslant t_1 \leqslant 0\}$. Let $t_0 = 0$. Now, since $4d_x F = \operatorname{Tr} g^{-1} dg$, each integrand in the above formula reduces to

$$\frac{\epsilon}{4} \operatorname{Tr} \left\{ g^{-1}(t_1^\star + t_l) \sum_{\{(i_m, j_m)\}} \prod_{m=l}^{n-1} c_{i_m j_m} \prod_{m=0}^{l-1} \tilde{g}_{i_m j_m}(t_1^\star + t_l - t_m) \left(\prod_{m=1}^{n} \partial_{i_m} \partial_{j_m} \right) h(t_1^\star + t_l) \right\}. \tag{3.74}$$

It is still not clear why such a formula would lead to a non-degenerate differential. However, let us assume that h has the following form in a neighbourhood of $\{x = 0\}$

$$h(x, s, dx) = \lambda(s) dx^2 \sum_{|\alpha| = 2n} u^\alpha x^\alpha + o(|x|^{2n})$$

where $u^\alpha = u_{\alpha_1} \dots u_{\alpha_{2n}}$, and likewise for x^α. We take u a constant vector in $\mathbb{R}^d$, and $\lambda(s)$ a smooth function, supported in $]0, t_1^\star[$. Formula (3.74) becomes

$$(c(u,u))^{n-l} \operatorname{Tr} g^{-1}(t_1^\star + t_l) \left\{ \lambda(t_1^\star + t_l) \prod_{m=0}^{l-1} \tilde{g}(t_1^\star + t_l - t_m)(u,u) \right\}.$$

Observe that these are nonnegative numbers. From those computations, we see that

$$\frac{a^n(g_\epsilon, [\overline{c}]) - a^n(g, [\overline{c}])}{a^0(g, [\overline{c}])} = \epsilon \int_0^{t_1^\star} \lambda(t) H(t) dt$$

where $H(t)$ is a function that does not vanish. This ends the proof for $n \geqslant 1$. □

3.5. Applications

We use simple Complex Analysis to locate zones without zeroes for φ. We also give some explicit examples corresponding to part (I) and (III) of the main theorem.

3.5.1. Complex Analysis and Dirichlet Series

Let $\lambda_0 < \lambda_1 < \dots < \lambda_k < \dots$ be positive real numbers. For $\delta > 0$, we let $\mathscr{D}(\delta, \lambda)$ be the set of Dirichlet series $L(s)$ whose abscissa of absolute convergence is $\leqslant \delta$, and

$$L(s) = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k^s}.$$

We let $\mathscr{D}^k(\delta, \lambda)$ be the set of $L \in \mathscr{D}(\delta, \lambda)$ such that $a_0 = \dots = a_{k-1} = 0$ and $a_k \neq 0$. For $0 < \lambda_\# \leqslant \lambda_0$, also consider $\mathscr{D}(\delta, \lambda, \lambda_\#)$ the set of holomorphic functions f on $\{\Re s > \delta\}$ such that there are $L_n \in \mathscr{D}(\delta, \lambda)$ with, for all $n \geqslant 0$

$$f(s) = L_0(s) + \frac{1}{s} L_1(s) + \dots + \frac{1}{s^n} L_n(s) + \mathcal{O}\left(\frac{1}{s^{n+1} \lambda_\#^s} \right).$$

We will denote (a_k^n) the coefficients of L_n. By taking notations coherent with the rest of the article, we have $s^{\kappa d/2} \varphi \in \mathscr{D}(\delta_g, \lambda, \lambda_\#)$. For $\delta' > \delta$ and $C > 0$, let

$$\Omega_{\delta', C} := \{ s \in \mathbb{C} \quad \Re s > \delta' \quad \Re s \leqslant C \log |\Im s| \}.$$

Lemma 3.5.1. *Let $f \in \mathscr{D}(\delta, \lambda, \lambda_\#)$ such that $L_0 \in \mathscr{D}^0(\delta, \lambda)$. Then there is a $\delta' > \delta$ such that for any constant $C > 0$, f has a finite number of zeroes in $\Omega_{\delta', C}$.*

In the special case where $\lambda_\# = \lambda_0$, we can take $\delta' > 0$ such that f has no *zeroes in*

$$\{ s \in \mathbb{C} \quad \Re s > \delta' \}.$$

Proof. We can write

$$L_0(s) = \frac{a_0^0}{\lambda_0^s} + \underbrace{\sum_{k=1}^{\infty} \frac{a_k^0}{\lambda_k^s}}_{:= \tilde{L}_0(s)}$$

There is a $\delta' > \delta$ such that whenever $\Re s > \delta'$,

$$|\tilde{L}_0(s)| \leqslant \frac{1}{3}\left|\frac{a_0^0}{\lambda_0^s}\right|$$

Take $N > 0$. Then for $|s|$ big enough — say $|s| > C_N$ — and for $\Re s > \delta'$,

$$\frac{1}{|s|}|L_1(s)| + \cdots + \frac{1}{|s|^{N-1}}|L_{N-1}(s)| \leqslant \frac{1}{3}\left|\frac{a_0^0}{\lambda_0^s}\right|.$$

We also find that

$$\Re s \leqslant \frac{N}{\log(\lambda_0/\lambda_\#)} \log|\Im s| + \mathcal{O}(1) \text{ and } |s| > C_N \implies \left|\frac{C'_N}{s^N \lambda_\#^s}\right| < \frac{1}{3}\left|\frac{a_0^0}{\lambda_0^s}\right|.$$

When $\lambda_0 = \lambda_\#$, the condition on $\Re s$ is void. When $\lambda_0 > \lambda_\#$, by taking $N \sim C\log(\lambda_0/\lambda_\#)$, we find that the zeroes of f in the region described in the lemma are actually in a bounded region of the plane. Since f is holomorphic, they have to be in finite number. □

We give another lemma:

Lemma 3.5.2. *Let $f \in \mathscr{D}(\delta, \lambda, \lambda_\#)$ be such that $L_0 \in \mathscr{D}^1(\delta, \lambda)$ and $L_1 \in \mathscr{D}^0(\delta, \lambda)$. Let*

$$\tilde{f}(s) := \frac{a_1^0}{\lambda_1^s} + \frac{a_0^1}{s\lambda_0^s}$$

There is a $\delta' > \delta$ such that for any constant $C > 0$, there is a mapping W from the zeroes of $\tilde{f}$ in $\Omega_{\delta',C}$ to the zeroes of f in $\Omega_{\delta',C}$, that only misses a finite number of zeroes of f, and such that

$$W(s) - s \underset{|s|\to\infty}{\to} 0.$$

A picture gives a better idea of the content of this abstract lemma. It is elementary to observe that the zeroes of $\tilde{f}$ are asymptotically distributed along a vertical log line $\Re s = a\log|\Im s| + b$, at intervals of lengths $\sim 2\pi(\log\lambda_1/\lambda_0)^{-1}$.

Remark 3.10. *Instead of assuming $a_0^0 = 0$ and $a_0^1, a_1^0 \neq 0$, we could have assumed a finite number of explicit cancellations and non-cancellations. In that case, it is likely that one could prove a similar lemma, with f having zeroes close to a finite (arbitrary) number of* log *lines instead of only one line. However, this leads to tedious computations that we did not carry out entirely.*

Proof. This is an application of Rouché's Theorem. We aim to give a good bound for $|f - \tilde{f}|$ on appropriate contours. To this end, we decompose

$$|f - \tilde{f}| \leqslant \left|L_0 - \frac{a_1^0}{\lambda_1^s}\right| + \frac{1}{|s|}\left|L_1 - \frac{a_0^1}{\lambda_0^s}\right| + \frac{1}{|s|^2}\left|L_2 + \cdots + \frac{1}{s^{n-2}}L_n\right| + \mathcal{O}\left(\frac{1}{|s|^{n-1}\lambda_\#^{\Re s}}\right) \quad (3.75)$$

For some $\delta' > \delta$, and for $\Re s > \delta'$,this gives

$$\leqslant C\left(\frac{1}{\lambda_2^{\Re s}} + \frac{1}{|s|\lambda_1^{\Re s}} + \frac{1}{|s|^2\lambda_0^{\Re s}} + \frac{1}{|s|^{n+1}\lambda_\#^{\Re s}}\right) \quad (3.76)$$

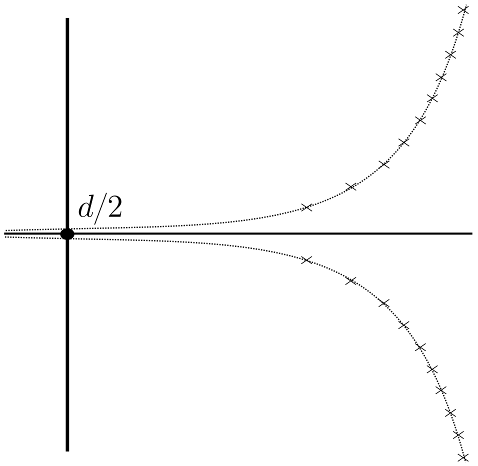

Figure 3.6: The zeroes of $\tilde{f}$.

where $C > 0$ is a constant. We can always choose δ' big enough so that

$$\frac{C}{\lambda_2^{\Re s}} \leqslant \frac{1}{2}|\tilde{f}|$$

for $\Re s = \delta'$ and $|\Im s|$ big enough. Then, on the vertical line $\Re s = \delta'$, for $|\Im s|$ big enough, $|f - \tilde{f}| < |\tilde{f}|$.

Now, on the line $\Re s = n \log |\Im s| (\log \lambda_0/\lambda_\#)^{-1}$, the 3 first terms of the RHS of equation (3.76) are very small in comparison to $\tilde{f}$. We can check that the last one is $\mathcal{O}(1/s)|\tilde{f}$ to see that on that line also, $|f - \tilde{f}| < |\tilde{f}|$.

Now, we observe that

$$\begin{aligned} |\tilde{f}| = \left|\frac{a_1^0}{\lambda_1^s}\right| + \left|\frac{a_0^1}{s\lambda_0^s}\right| \\ \iff \Im s \log \frac{\lambda_0}{\lambda_1} + \arg s + \arg \frac{a_1^0}{a_0^1} \in 2\pi\mathbb{Z}. \end{aligned}$$

Since in the region $\Omega = \{\delta' \leqslant \Re s \leqslant n \log |\Im s|(\log \lambda_0/\lambda_\#)^{-1}\}$, $\arg s = \pi/2 + \mathcal{O}(\log|s|/|s|)$, we deduce that there is a constant $C > 0$ such that,

$$2|\tilde{f}| \geqslant \left|\frac{a_1^0}{\lambda_1^s}\right| + \left|\frac{a_0^1}{s\lambda_0^s}\right|$$

on each line $\Im s = C + 2\pi k(\log \frac{\lambda_1}{\lambda_0})^{-1}$, $k \in \mathbb{N}$, in Ω. One can check that this implies that on each of those horizontal lines, $|f - \tilde{f}| < |\tilde{f}|$.

Now, the zeroes of $\tilde{f}$ are located on the line

$$a_0^1 |s| \lambda_1^{\Re s} = \lambda_0^{\Re s} a_1^0.$$

At distance $\mathcal{O}(1)$ of that line, one can see that the RHS in (3.76) is bounded by

$$\mathcal{O}(|s|^{-\alpha}) \left|\frac{a_1^0}{\lambda_1^s}\right|$$

for some $\alpha > 0$. The proof of the lemma will be complete if we can find some circles C_n around the zeroes s_n of $\tilde{f}$, whose radii r_n shrink, but such that on C_n,

$$|\tilde{f}| >> |s_n|^{-\alpha} \left| \frac{a_1^0}{\lambda_1^{s_n}} \right|.$$

Actually, this kind of estimate is true on the circles C_n centered at s_n of radius r_n, as long $r_n \to 0$ with $r_n >> |s_n|^{-\alpha}$. □

Now,

Lemma 3.5.3. *There are different situations.*

1. *When there is only* 1 *cusp, we always have* $L_0 \in \mathscr{D}^0(\delta, \lambda)$.
2. *In general, the set of* $g \in \mathcal{G}(M)$ *such that* $L_0 \in \mathscr{D}^0(\delta, \lambda)$ *is open and dense in* C^2 *topology.*
3. *There are examples of* hyperbolic *cusp surfaces with* $L_0 \in \mathscr{D}^1(\delta, \lambda)$.
4. *There are examples of* hyperbolic *cusp surfaces* M *that satisfy the following. First,* $L_0 \in \mathscr{D}^0(\delta, \lambda)$. *Second, there is an open set* $U \subset\subset M$ *such that for any cusps* Z_i, Z_j, $d(U, Z_i) + d(U, Z_j) \geqslant \mathcal{T}_{ij}^0 + \log a_i + \log a_j$. *Then* $\lambda_\# = \lambda$ *for all the metrics* $g \in \mathcal{G}_U(M)$ *(the metrics with variable curvature supported in* U*).*

Lemmas 3.5.1, 3.5.2 and 3.5.3 can be combined to prove theorem 3.1. Let us first prove lemma 3.5.3.

Proof. When $\kappa = 1$, $\varphi = \phi_{11}$. From lemma 3.3.4, we see that a_0^0 is a sum of positive terms over the set of scattered geodesics whose sojourn time is $\mathcal{T}_{11}^0$, hence it cannot vanish.

In the general case, the openness property of lemma 3.4.6 shows that for an open and dense set of $g \in \mathcal{G}(M)$ for the C^2 topology, the smallest element $\mathcal{T}^0$ of the set of sojourn cycles is simple. That implies that $a_0^0 \neq 0$.

For the third part of the lemma, an example will be constructed in section 3.5.2.2.

For the last part, an example will be given in section 3.5.2.1. The conclusion $\lambda_\# = \lambda_0$ is a consequence of the discussion just before theorem 3.5 □

Proof of theorem 3.1. We can list the cases

1. Consider the hyperbolic surface described in lemma 3.5.3(4). For such a surface, for all $g \in \mathcal{G}_U(M)$, we have $L_0 \in \mathscr{D}^0(\delta, \lambda)$, and $\lambda_\# = \lambda_0$. We can apply the special case of lemma 3.5.1, to prove part (I).
2. For all manifolds with one cusp only, $L_0 \in \mathscr{D}^0(\delta, \lambda)$ so we can apply the general case of lemma 3.5.1.
3. When there is more than one cusp, case (2) of lemma 3.5.3 and lemma 3.5.1 lead to part (II) of theorem 3.1.
4. The example in case (3) of lemma 3.5.3 can be perturbed, preserving the condition $L_0 \in \mathscr{D}^1(\delta, \lambda)$, and with $L_1 \in \mathscr{D}^0(\delta, \lambda)$, according to lemma 3.4.6. We can then apply lemma 3.5.2 to prove part (III).

5. Finally, we can adapt the proof of lemma 3.5.1 to show that whenever at least one L_i is not the zero function, the conclusions of lemma 3.5.1 apply, if we replace "for all constant $C > 0$" by "for some constant $C > 0$". This proves part (IV).

□

3.5.2. Two examples

In this last section, we construct explicit hyperbolic examples that satisfy the conditions given in lemma 3.5.3.

3.5.2.1. An example with one cusp

Here, we construct a surface with one cusp, such that there are parts of the surface that are *far* from the cusp, in the appropriate sense.

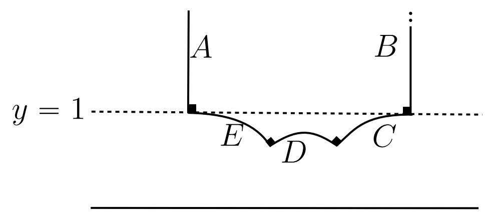

Figure 3.7: symmetric pentagon with an ideal vertex.

Topologically, we are looking at the most simple cusp surface: a punctured torus. It can be obtained explicitely by glueing two hyperbolic pentagons. Consider two copies of the pentagon in figure 3.7. Glueing sides $A \leftrightarrow B'$, $B \leftrightarrow A'$, $D \leftrightarrow D'$, $C \leftrightarrow E$ and $C' \leftrightarrow E'$, we obtain a punctured torus, that we call (M, g).

The scattered geodesic c_0 with the smallest sojourn time corresponds to the sides AB' and BA'. Its sojourn time is 0, i.e $\mathcal{T}^0 = 0$. However, the set U of points that are strictly below the line $\{y = 1\}$ is non empty (and open). This is the example in 4) in lemma 3.5.3

3.5.2.2. An example with 2 cusps

Now, we aim to construct an example of surface with two cusps (M, g) such that $L_0 \in \mathscr{D}^1$. We consider a two-punctured torus.

As in the previous example, we will glue pentagons. Only this time we glue 4 identical pentagons a, b, c, d, and they will not be symmetrical — see figure 3.8.

The cusp corresponding to pentagons a and b will be called cusp Z_1, and the other one, corresponding to pentagons c and d will be cusp Z_2. We obtain a surface (M, g_ℓ) with two cusps, depending on the hyperbolic length ℓ. To ensure the normalization condition that the volume of a projected horosphere at height y is y^{-1}, we have to take

$$y_0 = \frac{1}{2} \frac{1}{\sqrt{2} + \sqrt{1 + e^{-2\ell}}}. \tag{3.77}$$

We number the geodesics from i to j ($i = 1, 2$, and $j = 1, 2$) by their sojourn time c_n^{ij} with $\mathcal{T}(c_0^{ij}) \leqslant \mathcal{T}(c_1^{ij}) \leqslant \dots$. Since a geodesic coming from a cusp has to go under

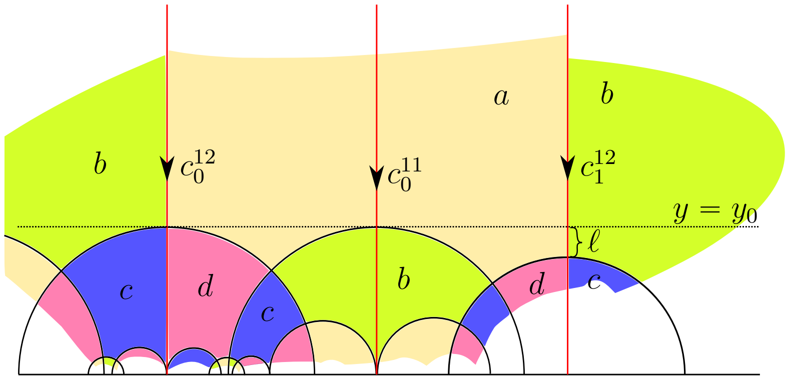

Figure 3.8: A tiling of the hyperbolic plane with pentagons.

the $y = y_0$ line to exit a pentagon, we see that actually, c_0^{11} and c_0^{12} are the geodesics designated in figure 3.8. Actually, we also get that :

$$-2\log y_0 = \mathcal{T}(c_0^{11}) = \mathcal{T}(c_0^{12}) < \mathcal{T}(c_1^{ij}), \quad i,j = 1,2.$$

This proves that $a_0^0 = 0$. Now, to obtain that $L_0 \in \mathscr{D}^1$, we need to show that the second shortest sojourn time is $2\ell - 2\log y_0$, and that it is *simple*. That is to say, c_1^{12} really *is* the curve drawn in figure 3.8, and the only other curves with sojourn time $\leqslant 2\ell - 2\log y_0$ are c_0^{11} and c_0^{12}.

In order to prove this, draw a line at height $y_0 e^{-2\ell}$. A scattered geodesic coming from cusp Z_1 can be lifted to $\mathbb{H}^2$ as a curve coming from ∞ in the pentagon a, that stays in the same pentagon as $y \leqslant y_0 e^{-2\ell}$. When ℓ is small enough, there are only 3 geodesics that satisfy such a property, and they are drawn on figure 3.8.

Chapitre 4

Propriétés de Comptage des résonances

Dans ce chapitre, je reproduis le contenu de deux articles traitant du comptage des résonances.
Dans la première partie, on trouvera le texte de [Bon14b], accepté pour publication dans le Journal of Spectral Theory. Il s'agit d'améliorer le théorème de Parnovski (WPR1) pour obtenir le théorème (WBR). Ce résultat ne nécessite pas d'hypothèse sur la géométrie de la partie compacte. On peut remarquer au passage que le théorème principal pourrait être généralisé en dimension supérieure si on pouvait généraliser le théorème (WP1).
La deuxième partie comprend les résultats d'une première version de [Bon15a]. Pour éviter les doublons, le texte a été réduit pour ne conserver que la preuve proprement dite de ses théorèmes. Il s'agit d'utiliser le théorème (1.50) pour démontrer des estimées de comptage aussi bonne qu'en courbure constante, généralisant (1.53). J'ai depuis trouvé comment améliorer encore ces résultats. La version que j'espère publier (et au moins mettre sur ArXiV), sera donc plus complète.

❈

4.1. Asymptotique de Weyl sans hypothèse sur la partie compacte

In this short note, we prove sharp bounds on resonance-counting functions for the Laplacian on finite volume surfaces with hyperbolic cusps. Let M be a complete non-compact surface, equipped with a Riemannian metric g. We assume that (M, g) can be decomposed as the union of a compact manifold with boundary and a finite number κ of hyperbolic cusps, each one being isometric to

$$(a, +\infty)_y \times \mathbb{S}^1_\theta \text{ with metric } \frac{dy^2 + d\theta^2}{y^2}$$

for some $a > 0$. The spectral properties of the Laplacian Δ_g were first studied by Selberg [Sel89a, Sel89b] and Lax-Phillips [LP76] in constant negative curvature, and by Colin-de-Verdière [CdV81, CdV83], Müller [Mül92], Parnovski [Par95] in the non-constant curvature setting.

On such surfaces, the resolvent $R(s) = (\Delta_g - s(1-s))^{-1}$ of the Laplacian admits a meromorphic extension from $\{\Re s > 1/2\}$ to $\mathbb{C}$ as an operator mapping L^2_{comp} to L^2_{loc} and the natural discrete spectral set for Δ_g is the set of poles denoted by

$$\text{Res}(M, g) \subset \{s \in \mathbb{C} | \ \Re s \leqslant 1/2\} \cup (1/2, 1].$$

The poles are called *resonances* and are counted with multiplicity $m(s)$ (the multiplicity $m(s)$ is defined below and corresponds, for all but finitely many resonances, to the rank of the residue of the resolvent at s). We shall recall in the next section how the set of resonances is built. To study their distribution in the complex plane, we define two counting functions :

$$N_{\text{Res}}(T) := \frac{1}{2} \sum_{\substack{s \in \text{Res}(M,g) \\ |s-1/2| \leqslant T}} m(s), \tag{4.1}$$

$$N_{\text{Res}}(T, \delta) := \sum_{\substack{s \in \text{Res}(M,g) \\ |s-1/2-iT| \leqslant \delta T}} m(s). \tag{4.2}$$

The first result on the resonance counting function was proved by Selberg [Sel89b, p. 25] for the special case of hyperbolic surfaces with finite volume : the following Weyl type asymptotic expansion holds as $T \to \infty$

$$N_{\text{Res}}(T) = \frac{\text{Vol}(M)}{4\pi} T^2 - \frac{\kappa}{\pi} T \log(T) + \frac{\kappa(1 - \log 2) - K}{\pi} T + O\Big(\frac{T}{\log(T)}\Big). \tag{4.3}$$

(K is an explicit geometric constant). In variable curvature, Müller gives a Weyl asymptotic [Mül92, Th. 1.3.a] of the form

$$N_{\text{Res}}(T) = \frac{\text{Vol}(M)}{4\pi} T^2 + o(T^2)$$

and this was improved by Parnovski [Par95] who showed that for all $\epsilon > 0$

$$N_{\text{Res}}(T) = \frac{\text{Vol}(M)}{4\pi} T^2 + O(T^{3/2+\epsilon}). \tag{4.4}$$

Parnovski's proof relies on a Weyl type asymptotic expansion involving the scattering phase $\mathcal{S}(T)$ (see next section for a precise definition) :

$$N_d(T) + \mathcal{S}(T) = \frac{\text{Vol}(M)}{4\pi} T^2 - \frac{\kappa}{\pi} T \ln T + O(T), \tag{4.5}$$

where κ is the number of cusps, and N_d is the counting function for the L^2 eigenvalues of Δ_g embedded in the continuous spectrum.

Using a Poisson formula proved by Müller [Mül92] and estimate (4.5), we are able to improve the result (4.4) of Parnovski :

Theorem 4.1. *For $T > 1$, and $0 \leqslant \delta \leqslant 1/2$, the following estimates hold*

$$N_{\mathrm{Res}}(T, \delta) = O(T^2\delta + T), \tag{4.6}$$

$$N_{\mathrm{Res}}(T) = \frac{\mathrm{Vol}(M)}{4\pi} T^2 + O(T^{3/2}). \tag{4.7}$$

In the first estimate with $\delta = 1/T$, the exponent in T is sharp in general, as can be seen from Selberg's result (4.3) and the additionnal estimate also from [Sel89b]

$$\sum_{\substack{s \in \mathrm{Res}(M,g) \\ 0 \leqslant \Im s \leqslant T}} \Re s - 1/2 = \frac{\kappa}{4\pi} T \log \frac{T}{\pi} - \frac{1}{2\pi} \left(\frac{\kappa}{2} + \log |c| \right) T + O(\log T), \tag{4.8}$$

where c is a constant depending on the surface, introduced by Selberg. Together these formulae imply that as $T \to \infty$

$$N_{\mathrm{Res}}(T, 1/T) = \frac{vol(M)}{2\pi} T + O\left(\frac{T}{\log T} \right). \tag{4.9}$$

In n-dimensional Euclidan scattering, upper bounds $O(T^{n-1})$ on the number of resonances in boxes of fixed size at frequency T were obtained by Petkov-Zworski [PZ99] using Breit-Wigner approximation and the scattering phase; our scheme of proof is inspired from their approach. Their result was extended to the case of non-compact perturbations of the Laplacian by Bony [Bon01]. In general, it is expected that the number of resonances in such boxes is controlled by the (fractal) dimension of the trapped set (see for example Zworski [Zwo99], Guillopé-Lin-Zworski [GLZ04], Sjöstrand-Zworski [SZ07], Datchev-Dyatlov [DD13]).

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4.1.1. Preliminaries

We start by recalling well-known facts on scattering theory on surfaces with cusps, and we refer to the article of Müller [Mül92] for details. Let (M, g) be a complete Riemannian surface that can be decomposed as follows:

$$M = M_0 \cup Z_1 \cup \cdots \cup Z_\kappa,$$

where M_0 is a compact surface with smooth boundary, and Z_j are hyperbolic cusps

$$Z_j \simeq (a_j, +\infty) \times \mathbb{S}^1, \quad j = 1 \ldots \kappa,$$

with $a_j > 0$ and the metric on Z_j in coordinates $(y, \theta) \in (a_j, +\infty) \times \mathbb{S}^1$ is

$$ds^2 = \frac{\mathrm{d}y^2 + \mathrm{d}\theta^2}{y^2}.$$

Notice that the surface has finite volume when equipped with this metric.

The non-negative Laplacian $-\Delta$ acting on $C_0^\infty(M)$ functions has a unique self-adjoint extension to $L^2(M)$ and its spectrum consists of

1. Absolutely continuous spectrum $\sigma_{ac} = [1/4, +\infty)$ with multiplicity κ (the number of cusps).

2. Discrete spectrum $\sigma_d = \{\lambda_0 = 0 < \lambda_1 \leqslant \cdots \leqslant \lambda_i \leqslant \ldots\}$, possibly finite, and which may contain embedded eigenvalues in the continuous spectrum. To $\lambda \in \sigma_d$, we associate a family of orthogonal eigenfunctions that generate its eigenspace $(u^i_\lambda)_{i=1\ldots d_\lambda} \in L^2(M) \cap C^\infty(M)$.

The generalized eigenfunctions associated to the absolutely continuous spectrum are the Eisenstein functions, $(E_j(x,s))_{i=1\ldots\kappa}$. Each E_j is a meromorphic family (in s) of smooth functions on M. Its poles are contained in the open half-plane $\{\Re s < 1/2\}$ or in $(1/2, 1]$. The Eisenstein functions are characterized by two properties :

1. $-\Delta E_j(., s) = s(1-s)E_j(., s)$

2. In the cusp Z_i, $i = 1\ldots\kappa$, the zeroth Fourier coefficient of E_j in the θ variable equals $\delta_{ij} y_i^s + \phi_{ij}(s) y_i^{1-s}$ where y_i denotes the y coordinate in the cusp Z_i and $\phi_{ij}(s)$ is a meromorphic function of s.

We can collect the *scattering coefficients* ϕ_{ij} in a meromorphic family of matrices, $\phi(s) = (\phi_{ij})_{ij}$ called *scattering matrix*. We denote its determinant by $\varphi(s) = \det\phi(s)$. Then the following identities hold

$$\phi(s)\phi(1-s) = Id, \quad \overline{\phi(s)} = \phi(\overline{s}), \quad \phi(s)^* = \phi(\overline{s}).$$

The line $\Re s = 1/2$ corresponds to the continuous spectrum. On that line, $\phi(s)$ is unitary, $\varphi(s)$ has modulus 1. We also define the scattering phase

$$\mathcal{S}(T) = -\frac{1}{2\pi}\int_0^T \frac{\varphi'}{\varphi}\left(\frac{1}{2} + it\right) \mathrm{d}t. \tag{4.10}$$

The set of poles of φ, ϕ and $(E_j)_{j=1\ldots\kappa}$ is the same, we call them them *scattering poles* and we shall denote $\mathcal{R}$ this set. It is contained in $\{\Re s < 1/2\} \cup (1/2, 1]$. The union of this set with the set of $s \in \mathbb{C}$ such that $s(1-s)$ is an L^2 eigenvalue, is called the resonance set, and denoted $\mathrm{Res}(M, g)$. Following [Mül92, pp.287], the multiplicities $m(s)$ are defined as :

1. If $\Re s \geqslant 1/2$, $s \neq 1/2$, $m(s)$ is the dimension of $\ker_{L^2}(-\Delta - s(1-s))$.

2. If $\Re s < 1/2$, $m(s)$ is the dimension of $\ker_{L^2}(-\Delta - s(1-s))$ minus the order of φ at s.

3. $m(1/2)$ equals $(\mathrm{Tr}(\phi(1/2)) + \kappa)/2$ plus twice the dimension of $\ker_{L^2}(-\Delta - 1/4)$.

For convenience, we define two counting functions for the discrete spectrum and the poles of φ:

$$N_d(T) := \sum_{|s_i - 1/2| \leqslant T} m(s_i), \tag{4.11}$$

$$N_{\mathcal{R}}(T) := \sum_{s \in \mathcal{R}, |s-1/2| \leqslant T} m(s), \tag{4.12}$$

so that

$$N_{\mathrm{Res}}(T) := \sum_{\substack{s \in \mathrm{Res}(M,g) \\ |s-1/2| \leqslant T}} m(s) = 2N_d(T) + N_{\mathcal{R}}(T). \tag{4.13}$$

4.1.2. Main observation

In this section, we obtain estimate for $N_{\mathrm{Res}}(T)$ in boxes at high frequency.

From the asymptotic expansion (4.5), we deduce that for $0 \leqslant \delta \leqslant 1/2$,

$$N_d(T+T\delta)-N_d(T-T\delta)+\mathcal{S}(T+T\delta)-\mathcal{S}(T-T\delta) = \frac{\mathrm{vol}(M)}{\pi}T^2\delta-\frac{2\kappa}{\pi}T\delta\ln T+O(T). \quad (4.14)$$

Next, we recall the Poisson formula for resonances proved by Müller [Mül92, Th. 3.32]

$$2\pi\mathcal{S}'(T) = \log\frac{1}{q} + \sum_{\rho\in\mathcal{R}} \frac{1-2\Re\rho}{(\Re\rho-1/2)^2+(\Im\rho-T)^2}. \quad (4.15)$$

where q is some positive constant (not necessarily < 1). Let $C > 1$, $0 < \epsilon < 1$ and

$$\Omega_{T,\delta} := \{s \in \mathbb{C};\ |s-1/2-iT| \leqslant T\delta/C \text{ and } 0 \leqslant 1/2-\Re s \leqslant \epsilon T\delta\}.$$

Then, for $s \in \Omega_{T,\delta}$,

$$\int_{[T-T\delta,T+T\delta]} \frac{1-2\Re s}{(\Re s-1/2)^2+(t-\Im s)^2}\mathrm{d}t = 2\left[\arctan\frac{t-\Im s}{1/2-\Re s}\right]_{T-T\delta}^{T+T\delta}.$$

The addition formula for arctan, with $x, y > 0$ and $xy > 1$ is given by

$$\arctan x + \arctan y = \pi + \arctan\frac{x+y}{1-xy}.$$

Thus

$$\begin{aligned}\int_{[T-T\delta,T+T\delta]} \frac{1-2\Re s}{(\Re s-1/2)^2+(t-\Im s)^2}\mathrm{d}t &= 2\pi - 2\arctan\frac{2T\delta(1/2-\Re s)}{T^2\delta^2-|s-1/2-iT|^2} \\ &\geqslant 2\pi - 2\arctan\tilde{C}\epsilon,\end{aligned}$$

where $\tilde{C}$ is set to be $2/(1-1/C^2)$. For ϵ small enough, this is bigger than, say, π.

Since all but a finite number of terms in (4.15) are positive, we have :

$$\mathcal{S}(T+T\delta)-\mathcal{S}(T-T\delta) \geqslant O(T\delta) + \sum_{\rho\in\mathcal{R}\cap\Omega_{T,\delta}} \frac{1}{2}.$$

Combining with (4.14), we deduce that

$$N_d(T+T\delta)-N_d(T-T\delta)+\#\mathcal{R}\cap\Omega_{T,\delta} = O(T^2\delta)+O(T)+O(T\delta).$$

This is the content of (4.6) in our main theorem.

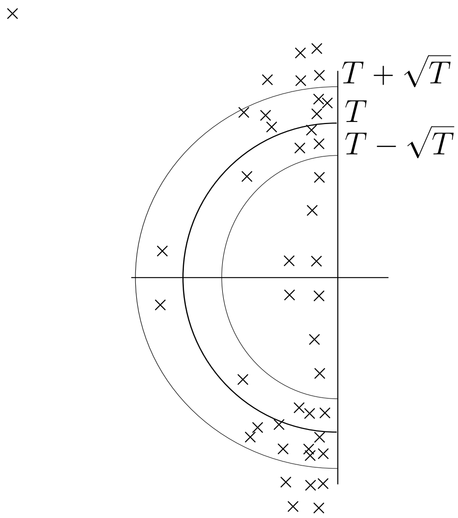

Figure 4.1: Dichotomy and counting.

4.1.3. Consequence

Now, we prove the second part of theorem 4.1. We will follow the method of Müller [Mül92, pp. 282], which is a global and quantitative version of the argument used in the previous section. Integrating the Poisson formula over $[-T, T]$, we relate the scattering phase asymptotics to the poles of ϕ. Using the arctan addition formula, we are left with the sum of $N_{\mathcal{R}}(T)$ and an expression with arctan's (equation (4.9) in [Mül92]) :

$$\mathcal{S}(T) =$$

$$\frac{1}{2}N_{\mathcal{R}}(T) + \frac{1}{2\pi}\sum_{\rho\in\mathcal{R},\Re\rho<1/2} \arctan\left[\frac{1-2\Re\rho}{|\rho-1/2|^2}T\left(1-\frac{T^2}{|\rho-1/2|^2}\right)^{-1}\right] + O(T). \tag{4.16}$$

The sum is then split between $\{1\}$ the poles in $\{|T-|\rho-1/2||>T^{1/2}\}$, and $\{2\}$, the others, as in figure 4.1. Müller proved that the sum $\{1\}$ is $O(T^{3/2})$. The sum $\{2\}$ can be bounded by

$$\frac{1}{4}(N_{\mathcal{R}}(T+\sqrt{T}) - N_{\mathcal{R}}(T-\sqrt{T})).$$

From [Mül92, Cor. 3.29], we also recall that

$$\sum_{\eta\in\mathcal{R},\eta\neq 1/2} m(\eta)\frac{1-2\Re\eta}{|\eta-1/2|^2} < \infty.$$

Consider the set $\tilde{\mathcal{R}} = \{\eta\in\mathcal{R};\ (2\Re\eta-1)^2 > \Im\eta,\ |\eta|>1\}$. On $\tilde{\Lambda}$, we have that $|\eta-1/2|^{1/2} \leqslant 1-2\Re\eta$, thus

$$\sum_{\eta\in\tilde{\mathcal{R}},\eta\neq 1/2} m(\eta)\frac{1}{|\eta-1/2|^{3/2}} < \infty.$$

If $\tilde{n}(T)$ is the counting function for $\tilde{\mathcal{R}}$, we deduce that

$$\sum_{k=1}^{\infty}\tilde{n}(k)\left[\frac{1}{k^{3/2}} - \frac{1}{(k+1)^{3/2}}\right] < \infty.$$

Since $\tilde{n}$ is non-decreasing, $\tilde{n}(k) = o(k^{3/2})$. Now,

$$N_{\mathcal{R}}(T - \sqrt{T}) - N_{\mathcal{R}}(T + \sqrt{T}) \leqslant \tilde{n}(T) + N_{\mathrm{Res}}(T, \sqrt{T}^{-1}) + N_{\mathrm{Res}}(-T, \sqrt{T}^{-1}).$$

This concludes the proof.

4.2. Estimée de comptage en courbure négative

The point of this article to give counting estimates on $\mathrm{Res}(M, g)$ when M is a surface with negative curvature.

When the curvature is constant, Selberg [Sel89b] proved that $\mathrm{Res}(M, g)$ is contained in a vertical strip $\{1/2 - \delta \leqslant \Re s \leqslant 1/2\}$, and the following estimates

Theorem (Selberg)**.** *There are some geometric constants a, K_1 and K_2 such that*

$$\sum_{|s| \leqslant T} 1 - 2\Re s = \frac{\kappa}{\pi} T \log \frac{T}{\pi} - \frac{1}{\pi}(\kappa + 2\log|a|)T + \mathcal{O}(\log T). \tag{4.17}$$

$$\#\{|s| \leqslant T\} = \frac{vol(M)}{4\pi} T^2 + K_1 T \log T + K_2 T + \mathcal{O}(T/\log T). \tag{4.18}$$

Given a cusp surface M, one can consider the set of metrics on M that give it a structure of cusp metrics. In this article, I prove a similar statement.

Theorem 4.2. *For a generic (in C^2 topology) cusp-metric on a cusp surface, of negative curvature, there exist constants $b > 1/2$ and C such that*

$$\#\{|s| \leqslant T, \Re s < 1 - b\} = \mathcal{O}(T), \tag{4.19}$$

$$\sum_{\substack{|s| \leqslant T \\ \Re s \geqslant 1-b}} 1 - 2\Re s = \frac{\kappa}{\pi} T \log T + CT + \mathcal{O}(\log T). \tag{4.20}$$

Additionally, when there is only one cusp, this is true for all negatively curved cusp metrics. The first formula shows that there are relatively few resonances that lie outside the strip $\{1 - b \leqslant \Re s \leqslant 1/2\}$.

Actually, I prove a slightly more general statement. Introduce $\mathscr{D}_b$ the set of Dirichlet series whose absolute abscissa of convergence is strictly smaller than b, and are bounded for $\Re s > b$. That is, series of the form

$$\sum_{k \geqslant 0} \frac{a_k}{\lambda_k^s} \text{ where } \lambda_k \in \mathbb{R} \tag{4.21}$$

that converge absolutely for $\Re s \geqslant b$. Also consider $\mathscr{D}_b^0$ those in $\mathscr{D}_b$ that tend to zero as $\Re s \to \infty$. Now define the following properties, valid for $\Re s = b$

$$\Re \frac{\varphi'}{\varphi} = -b\ell_0 + \Re \tilde{L}_0 + \mathcal{O}\left(\frac{1}{s}\right) \text{ where } \ell_0 \text{ is some positive constant, and } \tilde{L}_0 \in \mathscr{D}_b^0. \tag{$\Diamond b$}$$

and

$$\log|\varphi(s)| = -\frac{\kappa}{2}\log|s| + \log K - b\ell_0 + \Re\tilde{L}_1 + \mathcal{O}\left(\frac{1}{s}\right) \text{ where } K > 0 \text{ and again } \tilde{L}_1 \in \mathscr{D}_b^0. \tag{$\Diamond\Diamond b$}$$

Then

Theorem 4.3. *Let M be a cusp surface.*

1. *Assume* $(\Diamond b)$ *for some* $b > 1/2$. *Then* (4.19) *holds*

2. *Assume* both $(\Diamond b)$ *and* $(\Diamond\Diamond b)$ *for some* $b > 1/2$. *Then* (4.20) *holds with* $C = (\ell_0 - \kappa - 2\log K)/\pi$.

This result is coherent with that of Selberg (4.17) because in the case of constant curvature, one can check that the same coefficients appear — $\log|a| = -\ell_0/2$ and $K = \sqrt{\pi}^{\kappa}$. Observe that the parameter ℓ_0 is the smallest sojourn cycle as introduced in (3.11). The result is valid for generic negatively curved cusp metrics for the following reason. In [Bon15b], I give a parametrix for the scattering determinant when the curvature is negative as

$$\varphi(s) = s^{-\kappa/2} \sum_{k\geqslant 0} \frac{a_k(s)}{\lambda_k^s} + \mathcal{O}(\lambda_\#^{-s} s^{-N}). \tag{4.22}$$

This parametrix converges in a half plane $\{\Re s > \delta_g\}$, for some $\delta_g > 1/2$ that is 1 in constant curvature. The a_k's have asymptotic expansions as $|s| \to \infty$:

$$a_k(s) \sim a_k^0 + \frac{1}{s} a_k^1 + \dots .$$

This implies that in vertical strips $\{b_1 < \Re s < b_2\}$ with $b_1 > \delta$, and as $|s| \to \infty$, φ behaves as a sum of Dirichlet series

$$\varphi \sim s^{-\kappa/2} \sum_{n\geqslant 0} s^{-n} L_n(s)$$

provided at least one a_k^n does *not* vanish. When there is only one cusp, no coefficient a_k^0 vanishes. However, when there are more than one cusp, I show that some coefficients can vanish. Hence, I introduce the two following conditions

$$\text{At least one coefficient } a_k^n \text{ does not vanish.} \tag{$*$}$$

and

$$\text{At least one coefficient } a_k^0 \text{ does not vanish.} \tag{$**$}$$

From lemma 3.5.3, we know that both $(*)$ and $(**)$ are satisfied for surfaces with one cusp, and for generic metrics in the C^2 sense on surfaces with an arbitrary number of cusps. Using the parametrix (4.22), observe that

Lemma 4.2.1. *Hypothesis* $(*)$ *implies* $(\Diamond b)$ *for all* b *large enough (in particular* $\geq \delta_g$*). Hypothesis* $(**)$ *implies both* $(\Diamond b)$ *and* $(\Diamond\Diamond b)$ *for all* b *large enough. In particular, for generic metrics on a cusp surface, or for surfaces with one cusp, the latter is valid.*

If only $(*)$ *holds, the constants have to be modified in* $(\Diamond\Diamond b)$.

This lemma is a direct consequence of the fact that for a given Dirichlet series, there is a half plane where it cannot vanish. I conjecture that $(**)$ is actually satisfied for all cusp metric of negative curvature.

In all that follows, we will choose some $b > \delta$ big enough, but the constants will not depend on that choice.

The reader familiar with the proof in Selberg will notice the similarity. The main difference in the proof with that of Selberg is that because of the remainder in (4.22), we cannot integrate on horizontal half-lines, but have to use rectangles between $\{\Re s = 1/2\}$

and $\{\Re s = b\}$. However, that difference is inessential. All the hard work was done in [Bon15b], and one can see the developments below as mere verifications.

As a last remark, observe that the main theorem actually holds on cusp *manifolds*, up to changing the constants in (4.20), since the results in [Bon15b] are valid in any dimension.

4.2.1. Some lemmas from complex analysis

First, let us give some abstract lemmas on zeros of holomorphic functions. Take F a function holomorphic in a neighbourhood of a half plane $\{\Re z \geqslant a\}$. All sums are over the zeros of F, denoted by $z = \beta + i\gamma$ — following Selberg's notations. When a zero is sitting on the boundary of the counting box, it is counted with half multiplicity.

Lemma 4.2.2 (Carleman'). *Let $b > a$, and $T > 0$, and assume that F does not vanish on $\{\Re z = b\}$. Then*

$$2\pi \sum_{\beta > b, |z-b| < T} \log \frac{T}{|z-b|} = \int_{-\pi/2}^{\pi/2} \log |F(b + Te^{i\theta})| d\theta - \pi \log |F(b)| + \int_{-T}^{T} \log \frac{T}{|t|} \Re \frac{F'(b+it)}{F(b+it)} dt. \tag{4.23}$$

Now, additionally assume that $a = 1/2$, that $|F| = 1$ on the axis $\{\Re s = 1/2\}$, and that F is real on the real axis.

Lemma 4.2.3 (Counting in big rectangles). *Let $b > 1/2$.*

$$2\pi \sum_{1/2 \leqslant \beta \leqslant b, 0 \leqslant \gamma \leqslant T} (T - \gamma)(\beta - 1/2) = \int_0^T \Re \frac{F'}{F}(b+it)(b-1/2)(T-t)dt + \int_{1/2}^{b} \log \frac{|F(x+iT)|}{|F(x)|}(x - 1/2)dx - \int_0^T \log |F(b+it)|(T-t)dt. \tag{4.24}$$

Lemma 4.2.4 (Counting in small rectangles). *Take $b > 1/2$, and $c > 0$.*

$$2\pi \sum_{\substack{-\pi/c \leqslant \gamma - T \leqslant \pi/c \\ 1/2 \leqslant \beta \leqslant b}} \cos(c(\gamma - T)) \sinh(c(\beta - 1/2)) = \int_{-\pi/c}^{\pi/c} \sinh(c(b-1/2)) \cos(ct) \Re \frac{F'}{F}(b + iT + it) dt - c \int_{-\pi/c}^{\pi/c} \cosh(c(b-1/2)) \cos(ct) \log |F(b+iT+it)| dt + c \int_{1/2}^{b} \log(|F(x+iT+i\pi/c)|.|F(x+iT-i\pi/c)|) \sinh(c(x-1/2)) dx. \tag{4.25}$$

Proof. These three counting lemmas are obtained by considering the fact that $\log|F|$ is a harmonic function where F does not vanish. Hence, if u is another harmonic function on some open set Ω, such that F does not vanish on $\partial\Omega$, by Stoke's theorem,

$$2\pi \sum_{z\in\Omega, F(z)=0} u(z) = \int_{\partial\Omega} u\partial_\nu \log|F| - \log|F|\partial_\nu u.$$

If F vanishes on the boundary of Ω, by removing small half disks around those zeros, one find that they are counted with multiplicity $1/2$, in a similar formula.

For 4.2.2, we consider $u(z) = -\log|z-b|/T$, and integrate on the boundary of the half-disk. For (4.24), we take $u(z) = (T - \Re z)(\Im z - 1/2)$, and finally $u(z) = \cos(c(\Re s - T))\sinh(c(\Im z - 1/2))$ for (4.25).

□

The estimates on counting in boxes are similar to equations (1.1) in [Sel89b], and lemma 14, p. 319 in [Sel89a]. The Carleman' lemma is reminiscent of the usual Carleman theorem [Tit58, §3.7]. Last of this section is

Lemma 4.2.5. *Let $L \in \mathscr{D}_b^0$ be real on $\mathbb{R}$. Then, as $T \to \infty$,*

$$\int_0^T \Re L(b+it)dt = \mathcal{O}(1). \tag{4.26}$$

Proof. Since L converges absolutely in the region we are considering, we can write

$$L(b+it) = \frac{c_0}{\lambda_0^{it}} + \frac{c_1}{\lambda_1^{it}} + \cdots + \frac{c_k}{\lambda_k^{it}} + \ldots \tag{4.27}$$

where the c_k's are real, the λ_k's are real, ordered, and strictly greater than 1, and the sum converges normally. So we can estimate

$$\int_0^T \Re L = \sum_k c_k \frac{\sin(T\log\lambda_k)}{\log\lambda_k}. \tag{4.28}$$

Since all λ_k's are bigger than $\lambda_0 > 1$, and since $\sum|c_k| < \infty$, we conclude. □

4.2.2. The Maass-Selberg relations

The following lemma is valid for any cusp surface (without any assumption of curvature).

Lemma 4.2.6 (Maass Selberg)**.** *Take $y > 0$ big enough. The scattering determinant satisfies*

$$|\varphi(\sigma+it)| \leqslant y^{\kappa(2\sigma-1)} \left(\sqrt{1 + \frac{(\sigma-1/2)^2}{t^2}} + \frac{\sigma - 1/2}{|t|}\right)^\kappa \text{ when } \sigma > 1/2. \tag{4.29}$$

Proof. To prove this, we have to introduce the Eisenstein series. For each cusps Z_i, $E_i(s)$ is a meromorphic family of smooth functions on M that satisfy

$$-\Delta E_i(s) = s(1-s)E_i(s). \tag{4.30}$$

Additionnally, the zeroth Fourier coefficient of $E_i(s)$ in cusp Z_j equals

$$f_{ij}(y,s) = \delta_{ij} y^s + \phi_{ij}(s) y^{1-s}. \tag{4.31}$$

We let $\phi(s)$ be the matrix $(\phi_{ij}(s))_{ij}$. Its determinant is the *scattering determinant* $\varphi(s)$ — by definition. We denote by $W(s,y)$ the matrix whose coefficients are the f_{ij}. Let $\Pi^*_{y_0}$ be the projection on functions whose zero Fourier mode vanishes for $\{y > y_0\}$ in all cusps. Then we define $G_i^y(s) = \Pi_y^* E_i(s)$. We set to prove the Maass-Selberg formula. Actually, the proof of constant curvature works out identically. We differentiate (4.30) with respect to s, and we use Stoke's formula again to obtain

$$(1-2s)\int G_i^y(s)\overline{G_j^y(s)} + 2i\Im(s(1-s))\int \partial_s G_i^y \overline{G_j^y(s)} = \sum_k \left[\partial_s f_{ik}\overline{\partial_y f_{jk}} - \overline{f_{jk}}\partial_y\partial_s f_{ik}\right]. \tag{4.32}$$

The sum in the RHS is the (ij) coefficient of the matrix

$$\partial_s W \partial_y W^* - \partial_s\partial_y W.W^* = \partial_s(W.\partial_y W^* - \partial_y W.W^*). \tag{4.33}$$

In the LHS, it is $\partial_s[2i\Im(s(1-s))\int G_i^y\overline{G_j^y}]$. We deduce that there is a anti-meromorphic matrix-valued function $A(s)$ such that

$$2i\Im(s(1-s))\int G_i^y\overline{G_j^y} = A(s) + W.\partial_y W^* - \partial_y W.W^*. \tag{4.34}$$

Let $V(s)$ be the matrix with coefficients $\int G_i^y\overline{G_j^y}$. Elementary computations give

$$2i(1-2\Re s)\Im s V(s) = A(s) + (2\Re s - 1)(y^{-2i\Im s}\phi - y^{2i\Im s}\phi^*) + 2i\Im s(y^{1-2\Re s}\phi\phi^* - y^{2\Re s - 1}). \tag{4.35}$$

We deduce that $A(s)$ vanishes on the unitary axis $2\Re s = 1$, and thus has to vanish identically.

$$V(s) = \frac{y^{2i\Im s}\phi^* - y^{-2i\Im s}\phi}{2i\Im s} + \frac{y^{2\Re s - 1} - y^{1-2\Re s}\phi\phi^*}{2\Re s - 1} \quad \text{for } \Re s > 1/2. \tag{Maass-Selberg Relation}$$

The matrix on the LHS is non-negative, so that as a hermitian matrix,

$$\phi\phi^* \leqslant y^{2(2\Re s-1)} + \frac{2\Re s - 1}{\Im s}\frac{y^{2s}\phi^* - y^{2\bar{s}}\phi}{2iy}. \tag{4.36}$$

We deduce that

$$\phi\phi^* \leqslant y^{2(2\Re s-1)}\left(\sqrt{1 + \left(\frac{\Re s - 1/2}{\Im s}\right)^2} + \frac{\Re s - 1/2}{|\Im s|}\right)^2 \tag{4.37}$$

This formula is true as long as we are still in the cusp, with constant curvature. That is, we cannot take y arbitrarily small. □

Observe that taking the limit $\Re s \to 1/2$, we find

$$V\left(\frac{1}{2}+it\right) = 2\log y \mathbb{1} + \frac{y^{2it}\phi^* - y^{-2it}\phi}{2it} - \frac{1}{2}(\phi'\phi^* + \phi\phi^{*\prime}).$$

Since ϕ is unitary on the unitary axis, $\phi'\phi^*$ is self-adjoint, and we recover the classical

$$\int G^y \overline{G^y}\left(\frac{1}{2}+it\right) = 2\kappa \log y - \frac{\varphi'}{\varphi}\left(\frac{1}{2}+it\right) + \mathrm{Tr}\, \frac{y^{2it}\phi^* - y^{-2it}\phi}{2it}. \tag{4.38}$$

We conclude this section by observing that in higher dimension, the proof is identical, replacing $s(1-s)$ by $s(d-s)$, and eventually, replacing $1-2s$ by $d-2s$.

4.2.3. Counting resonances far from the spectrum

In this section, we assume only hypothesis $(\lozenge b)$ for some $b > 1/2$. We deduce from lemma 4.2.2 applied to φ that

$$2\pi \sum_{\Re s > b, |s-b|<T} \log \frac{T}{|s-b|} = \int_{-T}^{T} \log \frac{T}{|t|} \Re \frac{\varphi'(b+it)}{\varphi(b+it)} dt + \int_{-\pi/2}^{\pi/2} \log |\varphi(b+Te^{i\theta})| d\theta - \pi \log |\varphi(b)|.$$

From the Maass-Selberg estimate (lemma 4.2.6), we deduce that the second term is $\mathcal{O}(T)$ (the log of $|\varphi|$ is bounded by a $\mathcal{O}(T)$ term, and a $\log \sin\theta$). The third is a constant, and the first one is $\mathcal{O}(T)$ by $(\lozenge b)$ (because $(\lozenge b)$ implies that $\Re\varphi'/\varphi$ is bounded on $\{\Re s = b\}$). Hence part (1) of theorem 4.3.

4.2.4. Counting in vertical strips.

Now, we assume both $(\lozenge b)$ and $(\lozenge\lozenge b)$ for some $b > 1/2$. Using Maass-Selberg again, we see that there is a constant $C > 0$ such that

$$\int_{1/2}^{b} \log |\varphi(x+iT)|(x-1/2) dx \leqslant C. \tag{4.39}$$

The counting in small rectangles enables us to also have a lower bound. We proceed as in Selberg [Sel89b, p.21] The LHS in (4.25) is always positive, so we write

$$\begin{aligned} c\int_{1/2}^{b} \log \Big\{ |\varphi(x+iT+i\pi/c)|.|\varphi(x+iT-i\pi/c)| \Big\} \sinh(c(x-1/2)) dx > \\ - \int_{-\pi/c}^{\pi/c} \sinh(c(b-1/2)) \cos(ct) \Re \frac{\varphi'}{\varphi}(b+iT+it) dt \\ + c\int_{-\pi/c}^{\pi/c} \cosh(c(b-1/2)) \cos(ct) \log |\varphi(b+iT+it)| dt \end{aligned}$$

Hypothesis $(\lozenge b)$ implies that the first term in the RHS is $\mathcal{O}(1)$. Then, $(\lozenge\lozenge b)$ implies that the second term is $\mathcal{O}(\log T)$. Using the upper bound (4.39) on $\log|\varphi(z)|$ given by Maass-Selberg, we find for some constant $C > 0$ depending on c,

$$\int_{1/2}^{b} \log \Big\{ |\varphi(x+iT+i\pi/c)| \Big\} \sinh(c(x-1/2)) dx > -C \log T.$$

Now, we use again the upper bound of φ: for some $c' > 0$, $|\varphi(x+iT+i\pi/c)| \leqslant 1/c'$. So that for some constant $C > 0$,

$$\int_{1/2}^{b} \log\left\{|\varphi(x+iT+i\pi/c)|c'\right\}(x-1/2)dx \geqslant C\int_{1/2}^{b} \log\left\{|\varphi(x+iT+i\pi/c)|c'\right\}\sinh(c(x-1/2))dx$$

We conclude that

$$\int_{1/2}^{b} \log|\varphi(x+iT)|(x-1/2)dx = \mathcal{O}(\log T). \tag{4.40}$$

Applying (4.25) again, we see that

$$\sum_{\substack{\beta\leqslant b\\ T\leqslant\gamma\leqslant T+1}} \beta - 1/2 = \mathcal{O}(\log T). \tag{4.41}$$

Now, we apply equation (4.24) for φ at T and at $T+1$, and we subtract the two equalities. We obtain

$$\begin{aligned} 2\pi \sum_{\substack{1/2\leqslant\beta\leqslant b\\ 0\leqslant\gamma\leqslant T}} (\beta-1/2) = &\int_0^T \Re\frac{\varphi'}{\varphi}(b+it)(b-1/2)dt \\ &+ \int_{1/2}^{b} \log\frac{|\varphi(x+iT+i)|}{|\varphi(x+iT)|}(x-1/2)dx \\ &- \int_0^T \log|\varphi(b+it)|dt. \\ &+ R \end{aligned} \tag{4.42}$$

where the remainder R is

$$\begin{aligned} R = &- \sum_{\substack{1/2\leqslant\beta\leqslant b\\ T\leqslant\gamma\leqslant T+1}} (T+1-\gamma)(\beta-1/2) \\ &+ \int_0^1 \Re\frac{\varphi'}{\varphi}(b+i(T+1-t))t(b-1/2)dt \\ &- \int_0^1 \log|\varphi(b+i(T+1-t))|tdt. \end{aligned}$$

In R, the first term is $\mathcal{O}(\log T)$ by (4.41). The second one is $\mathcal{O}(1)$ by ($\Diamond b$), and the last one is $C\log T + \mathcal{O}(1)$ by ($\Diamond\Diamond b$).

In the RHS of (4.42), equation (4.40) proves that the second term is $\mathcal{O}(\log T)$. For the first and third terms, we can use lemma 4.2.5, ($\Diamond b$) and ($\Diamond\Diamond b$) to see that

$$\pi \sum_{1/2\leqslant\beta<b, |\gamma|\leqslant T} \beta - 1/2 = -\ell_0 T(b-1/2) + \frac{\kappa}{2}T\log T - \frac{\kappa}{2}T - T\log K + b\ell_0 T + \mathcal{O}(\log T).$$

and the conclusion

$$\sum_{1/2\leqslant\beta<b, |\gamma|\leqslant T} \beta - 1/2 = \frac{\kappa}{2\pi}T\log T + \frac{1}{2\pi}(\ell_0 - \kappa - 2\log K)T + \mathcal{O}(\log T).$$

□

Chapitre 5

Une applications de la paramétrice

Ce chapitre est dédiée à la preuve d'une estimation asymptotique de certaines distributions qu'il faudrait comprendre plus en détail pour arriver à montrer une formule de trace sur les états résonants. Une telle formule serait un premier pas possible dans la résolution de la conjecture d'ergodicité quantique pour les états résonnants de Zworski. La résolution de cette conjecture semble aujourd'hui hors d'atteinte.

❈

5.1. Vers une formule de Trace

La preuve de l'Ergodicité Quantique pour le Laplacien sur une variété compacte est désormais classique. Comme on l'a vu dans la section 2.3, celle de l'Ergodicité Quantique pour les séries d'Eisenstein est relativement élémentaire une fois que l'on dispose des relations de Maass-Selberg, ou autrement dit d'une formule pour la 0-trace du Laplacien (voir dans l'introduction, 1.26). Dans les deux cas, on a recours de façon crucial à une formule de trace.

À ce jour, à ma connaissance, il n'existe pas de résultat d'Ergodicité Quantique qui serait formulée à partir des états résonnants, et qui concernerait toute la famille des états résonnants d'un certain opérateur dont le symbole principal aurait un flot hamiltonien ergodique. Il n'est d'ailleurs même pas clair de savoir ce qu'un tel résultat pourrait être. Tentons d'expliquer pourquoi.

On considère $\sigma \in C_c^\infty(T^*M)$, supporté dans les couches d'énergies $\{|p - 1/2| \leqslant \epsilon\}$. Comme les preuves qui fonctionnent utilisent une formule de trace, on part de cela. La trace de la restriction au spectre continu donne

$$\operatorname{Tr}_{ac} \operatorname{Op}(\sigma) = \frac{1}{4\pi} \int_{\mathbb{R}} \left\langle \operatorname{Op}(\sigma) E\left(\frac{d}{2} + it\right), E\left(\frac{d}{2} + it\right) \right\rangle dt. \tag{5.1}$$

Cette quantité est bien définie car σ est à support compact (voir le premier article). Avant de se lancer plus avant dans les calculs, on remarque que d'après la localisation des fonctions propres 2.3.1, la partie de l'intégrale qui donne une contribution non négligeable correspond à $\{||h|t| - 1| \leqslant 2\epsilon\}$.

On considère la fonction $\mathrm{Tr}_\sigma(d/2+it) := \langle \mathrm{Op}(\sigma)E(d/2+it), E(d/2+it)\rangle$. On peut étendre cette fonction en une fonction méromorphe en dehors de l'axe unitaire par

$$\mathrm{Tr}_\sigma(s) = \langle \mathrm{Op}(\sigma)E(s), E(d-\overline{s})\rangle. \tag{5.2}$$

Étant donné que $E(s) = \phi(s)E(d-s)$, on trouve que

$$\mathrm{Tr}_\sigma(s) = \sum_{i,j=1\dots\kappa} \phi_{ij}(d-s)\int_M E_j(s)\times \mathrm{Op}(\sigma)E_i(s). \tag{5.3}$$

On a alors l'espoir de pouvoir appliquer un argument de déformation de contour pour montrer que la trace de $\mathrm{Op}(\sigma)$ s'écrit comme une somme sur les résonances. Naïvement, on considère un contour rectangulaire dont deux sommets opposés sont $d/2+i(1-2\epsilon)/h$ et $\delta+i(1+2\epsilon)/h$, de sorte que $\delta > d/2$ soit suffisemment grand (par exemple, au delà de la bande où presque toutes les résonnances sont localisées).

Malheureusement, il n'est pas évident de montrer que l'intégrale de $\mathrm{Tr}_\sigma(s)$ le long des côtés horizontaux est contrôlée. Pour ce faire, il faudrait montrer que dans la zone $\{d/2 < \Re s < \delta_g, |\Im s - 1/h| \leqslant \epsilon/h\}$, zone où φ a de nombreux zéros, φ prend aussi des valeurs suffisemment grandes. C'est un résultat qui semble pour le moment hors d'atteinte.

Nous allons nous contenter de démontrer

Théorème 5.1. *Quand $s = \eta + i\lambda/h$ avec $\eta > \delta_g$,*

$$\varphi(s)\,\mathrm{Tr}_\sigma(s) = \sum_{\{\gamma_1,\dots,\gamma_\kappa\}} (-1)^{\epsilon\{\gamma\}} \exp\left(\sum_i \int_{\gamma_i} V_0 - sT(\gamma_i)\right)\left\{\sum_i \int_{-\lambda\gamma_i} \sigma + \mathcal{O}(h)\right\}. \tag{5.4}$$

où la somme est entendue sur les cycles géodésiques, comme définis après l'équation (3.11)*, et $\epsilon\{\gamma\}$ est la signature de la permutation associée au cycle $\{\gamma\} = \{\gamma_1,\dots,\gamma_\kappa\}$. Le reste est uniforme en λ tant que λ reste dans un compact de $\mathbb{R}_+^*$.*

Rappelons que pour les mêmes valeurs de s, d'après le théorème 3.5,

$$\varphi'(s) = \sum_{\{\gamma_1,\dots,\gamma_\kappa\}} (-1)^{\epsilon\{\gamma\}} \exp\left(\sum_i \int_{\gamma_i} V_0 - sT(\gamma_i)\right)\left\{\sum_i T(\gamma_i) + \mathcal{O}(h)\right\} \tag{5.5}$$

La similitude entre les deux expressions suggère que le théorème d'Ergodicité Quantique puisse être relié à des propriétés d'équidistribution des géodésiques diffusées. Dans l'article [PP13], Parkkonen et Paulin démontrent justement un tel résultat d'équidistribution (c'est leur théorème 2). Il est conditionnel au fait que la mesure de Gibbs (voir [PPS12]) d'un certain potentiel soit finie. Dans le cas qui nous intéresse, il s'agirait de déterminer la finitude de la mesure de Gibbs du potentiel V_0.

Profitons-en pour faire une remarque. Autant dans le cas de la courbure constante ou le cas des variétés compactes de courbure négative, il est connu depuis longtemps que la mesure de Gibbs du Jacobien instable est la mesure de Liouville, autant cela ne semble par avoir été démontré dans le cas de la courbure variable. Ce serait la conséquence d'une inégalité de Ruelle. Quand M est une variété compacte de courbure négative et φ_t son flot géodésique, cette inégalité est [Rue78]

$$h_\mu(\varphi_t) \leqslant \int_{SM} \lambda^+ d\mu \tag{5.6}$$

Felipe Riquelme, en thèse avec Barbara Schapira, semble avoir obtenu des résultats dans cette direction, qui ne sont malheureusement pas parus à ce jour.

Le théorème 5.1 peut se déduire du théorème 3.4 et du lemme 5.1.1 en utilisant la formule de la comatrice. Posons

$$\mathrm{Tr}_\sigma^{ij}(s) := \int_M E_j(s) \times \mathrm{Op}(\sigma) E_i(s). \tag{5.7}$$

Ces objets sont des distributions en σ. Contrairement aux distributions de Wigner étudiées dans la partie 2.2.2, elles ne convergent pas à priori vers des mesures invariantes. De fait, nous allons voir que quand $\Im s \to \pm\infty$, elle ne convergent même pas du tout, ce qui ne nous empêche pas d'en donner une asymptotique :

Lemme 5.1.1. *Soit (M, g) une variété à pointe de courbure strictement négative, et δ_g l'exposant de convergence de sa paramétrice. Alors, dès que $\eta > \delta_g$, quand $h \to \pm 0$,*

$$\mathrm{Tr}_\sigma^{ij}\left(\eta + i\frac{\lambda}{h}\right) = (\pi h)^{d/2} \left\{ \sum_{\gamma \in \pi_1^{ij}(M,g)} e^{-(\eta + i\lambda/h)T(\gamma)} \exp\left\{ \int_\gamma V_0 \right\} \int_{-\lambda\gamma} \sigma \right\} + \mathcal{O}(h). \tag{5.8}$$

Comme nous allons le voir, on a même une asymptotique à tout ordre en puissance de h, avec un reste en $\mathcal{O}(h^N)$. L'estimation est uniforme en λ tant que $|\lambda|$ reste dans un compact de $\mathbb{R}^{+*}$.

5.2. Un calcul dans la zone de convergence

Cette dernière partie est consacrée à la preuve du lemme 5.1.1. Avant d'en arriver à la preuve proprement dit, nous avons besoin de préciser un peu la construction de la quantification présentée dans la section 2.2.1.1.

5.2.1. Quantification adaptée au revêtement

Soit σ un symbole sur M, et $f \in C_c^\infty(M)$. On peut relever ces fonctions en des fonctions invariantes par Γ sur $\widetilde{M}$. La quantification sur une pointe Z était construite de sorte qu'il y avait une quantication $\widetilde{\mathrm{Op}}$ sur $\mathbb{H}^{d+1}$ telle que dans Z,

$$\widetilde{\mathrm{Op}(\sigma)f} = \widetilde{\mathrm{Op}}(\widetilde{\sigma})\widetilde{f}. \tag{5.9}$$

Cette quantification vérifiait donc la propriété d'équivariance $(\widetilde{\mathrm{Op}}(\sigma)f) \circ \gamma = \widetilde{\mathrm{Op}}(\gamma^* \sigma) f \circ \gamma$.

On peut construire sur $\widetilde{M}$ une quantification $\widetilde{\mathrm{Op}}$ qui vérifie la même propriété d'équivariance, et qui correspond de la même façon à la quantification Op sur M. Donnons-nous une partition de l'unité $\chi_0, \dots \chi_\kappa$ sur M de sorte que

$$\chi_0^2 + \cdots + \chi_\kappa^2 = 1 \tag{5.10}$$

et χ_i est supporté dans Z_i pour $i = 1, \dots, \kappa$, et vaut 1 pour $y \geqslant 2a_0 = 2 \max a_i$. On note M_0' la variété compacte à bords $\{y_M \leqslant 2a\}$. Elle est le support de χ_0. On note $\widetilde{M}_0'$ le fermé qui la revêt dans $\widetilde{M}$.

En découpant le long d'hypersurfaces , on peut obtenir un domaine fondamental compact $\mathcal{D}_0'$ pour l'action de Γ sur $\widetilde{M}_0'$ qui est l'adhérence de son intérieur, et qui est une variété à bord et à coins. En convolant dans des cartes l'indicatrice de $\mathcal{D}_0'$ avec une fonction C_c^∞, on obtient une fonction χ^2, C^∞ à support compact dans $\widetilde{M}$, dont le support rencontre un nombre fini de $\gamma\mathcal{D}_0'$, et tel que sur $\widetilde{M}$,

$$\chi_0^2 = \sum_{\gamma\in\Gamma} \chi_0^2 (\chi\circ\gamma)^2 \tag{5.11}$$

Le support de χ a un voisinage ouvert U simplement connexe. On peut donc trouver une carte Ψ qui l'envoie sur un ouvert simplement connexe V de $\mathbb{R}^n$. On se donne une quantification $\mathrm{Op}_{\mathbb{R}^n}$ sur $\mathbb{R}^n$, et on la tire en arrière par Ψ pour obtenir une quantification Op_U sur U. On définit une quantification sur $\widetilde{M}_0'$ par la formule standard

$$\widetilde{\mathrm{Op}}(\sigma) = \sum_{\gamma\in\Gamma} \chi\circ\gamma \left\{(\gamma^* \mathrm{Op}_U)(\sigma)\right\} \chi\circ\gamma. \tag{5.12}$$

On peut vérifier que pour $\gamma_0 \in \Gamma$,

$$\widetilde{\mathrm{Op}}(\gamma_0^*\sigma)(f\circ\gamma_0) = \sum_{\gamma\in\Gamma} \chi\circ\gamma \left\{(\gamma^* \mathrm{Op}_U)(\gamma_0^*\sigma)\right\} (\chi\circ\gamma \times f\circ\gamma_0) \tag{5.13}$$

$$= \sum_{\gamma\in\Gamma} \chi\circ\gamma \left\{\mathrm{Op}_U(\gamma_*\gamma_0^*\sigma)(\chi\times f\circ\gamma_0\circ\gamma^{-1})\right\}\circ\gamma \tag{5.14}$$

$$= \sum_{\gamma\in\Gamma} \chi\circ\gamma\circ\gamma_0 \left\{\mathrm{Op}_U(\gamma_*\sigma)(\chi\times f\circ\gamma^{-1})\right\}\circ\gamma\circ\gamma_0 \tag{5.15}$$

$$= \left\{\widetilde{\mathrm{Op}}(\sigma) f\right\}\circ\gamma_0. \tag{5.16}$$

En particulier, si on choisit σ et f invariants par Γ, on obtient une fonction invariante par Γ, ce qui montre que $\widetilde{\mathrm{Op}}$ définit une quantification Op sur M_0'. On peut alors poser sur M

$$\mathrm{Op}(\sigma) = \sum_{i=0}^{\kappa} \chi_i \,\mathrm{Op}(\sigma_{|Z_i})\chi_i. \tag{5.17}$$

En choisissant dans chaque horoboule $B(p, a_0)$, avec $p \in \Lambda^{par}$, de quantifier par la quantification Op sur $\mathbb{H}^{d+1}$ utilisée pour construire la quantification sur les pointes, on peut recoller tous ces morceaux pour obtenir une quantification $\widetilde{\mathrm{Op}}$ sur $\widetilde{M}$ toute entière qui vérifie bien la propriété d'équivariance.

Remarque 5.1. *Si on voulait obtenir une quantification de Weyl, il faudrait choisir une carte qui envoie le volume riemannien de la variété $\widetilde{M}$ sur la mesure de Lebesgue, ce qui est toujours possible, et tirer en arrière la quantification de Weyl sur $\mathbb{R}^n$.*

C'est cette quantification de Weyl que nous utiliserons dans la prochaine section. De plus, il sera utile de supposer que nous avons tronqué le noyau de Op à une distance $\epsilon_0 > 0$ de la diagonale. Étant donné la proposition 2.1.15, on peut faire cela de façon tout à fait légale. La distance ϵ_0 sera fixée tout au long de la preuve.

5.2.2. Retour à la preuve de 5.1.1

D'abord, on remarque Tr_σ^{ij} est bien défini car σ est à support compact. Dans ce qui suit, on note $s = \eta + i\lambda/h$, avec $||\lambda| - 1| \leqslant 2\epsilon$. On va utiliser les résultats et notations de la partie 3. Dans les restes, tous les termes en b_i^s seront remplacés par 1 car ici la partie réelle de s, η, est fixe, et on ne cherche pas à faire tendre η vers l'infini.

Utilisons la paramétrice pour E_i, (théorème 3.3). On se donne un point p dans le bord de $\widetilde{M}$, qui représente la pointe Z_i. Pour tout $x \in M$, on se donne un relêvement $\tilde{x} \in \widetilde{M}$. Alors :

$$E_i(s,x) = \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} P_N(\gamma\tilde{x}, p, s) + \mathcal{O}_{L^2}(h^N).$$

D'après les remarques de 5.2.1 de Op sur M, on a donc

$$(\mathrm{Op}(\sigma)E_i)(s,x) = \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} \left\{\mathrm{Op}(\sigma)P_N(.,p,s)\right\}(\gamma\tilde{x}) + \mathrm{Op}(\sigma).\mathcal{O}(h^N).$$

En prenant q un point représentant la pointe Z_j dans le bord de $\widetilde{M}$, on calcule :

$$E_j \times \mathrm{Op}(\sigma)E_i(s,x) = \sum_{[\gamma']\in\Gamma_q\backslash\Gamma}\ \sum_{[\gamma]\in\Gamma_p\backslash\Gamma} P_N(\gamma'\tilde{x}, q, s) \times \left\{\mathrm{Op}(\sigma)P_N(.,p,s)\right\}(\gamma\tilde{x}) + R \quad (5.18)$$

où R est un reste

$$R = \mathrm{Op}(\sigma)E_i \times \mathcal{O}_{L^2}(h^N) + E_j \times \mathrm{Op}(\sigma)\mathcal{O}_{L^2}(h^N) + \mathrm{Op}(\sigma)\mathcal{O}_{L^2}(h^N) \times \mathcal{O}_{L^2}(h^N)$$

C'est $\mathcal{O}(h^N)$ dans L^1, donc on peut oublier R et se concentrer sur l'autre terme dans (5.18). En procédant comme dans la preuve du lemme 3.3.1, on trouve que ce terme principal est égal à

$$\sum_{[\gamma]\in\pi_1^{ij}(M)} \int_{\widetilde{M}} P_N(\gamma.,q,s)\left\{\mathrm{Op}(\sigma)P_N\right\}(.,p,s) + \delta_{ij}\int_{\Gamma_p\backslash\widetilde{M}} P_N(.,p,s)\left\{\mathrm{Op}(\sigma)P_N\right\}(.,p,s). \quad (5.19)$$

On peut alors procéder de la même façon que pour la démonstration du théorème 3.4, en étudiant chaque terme de la somme. Avant d'aller plus loin, on a besoin du lemme suivant :

Lemme 5.2.1. *Si $\sigma \in S^n(\widetilde{M})$ est supporté au dessus d'un ouvert de $\widetilde{M}$ qui se projette dans M sur un ouvert relativement compact, et si $p \in \Lambda_{par}^i$, alors*

$$\left\{\mathrm{Op}(\sigma)P_N\right\}(x,p,s) = e^{-sG_p(x)}\tilde{J}_p f_{\sigma,p}^N(x,s), \quad (5.20)$$

où $f_{\sigma,p}^N(x,s) = \sigma(x, -\lambda d_x G_p) + \mathcal{O}(h)$ où $\lambda = h\Im s$. En fait, $f_{\sigma,p}^N$ a une expansion semi-classique à tout ordre,

$$f_{\sigma,p}^N = \sigma(x, -\lambda d_x G_p) + h f_{\sigma,p,1} + \cdots + h^{N-1} f_{\sigma,p,N-1} + \mathcal{O}(h^N(1 + G_p^+(x)^{2N})). \quad (5.21)$$

Les $f_{\sigma,p,n}$ font intervenir les dérivées de σ, et sont $\mathcal{O}(1 + G_p^+(x)^n)$. De plus, le reste n'est supporté que pour x à une distance bornée du support de σ (distance qui ne dépend pas de σ, mais seulement de la quantification).

Ce lemme ne fait que traduire le fait qu'un opérateur pseudo-différentiel doit préserver les états lagrangiens.

Démonstration. Comme on a imposé que Op soit uniformément proprement supportée, et que σ est à support compact, en écrivant les expressions en cartes, on se retrouve directement à faire des calculs dans un compact de $\mathbb{R}^{2d+2}$. On veut appliquer le théorème [Hör03, 7.7.5]. On écrit l'intégrand comme

$$\begin{aligned} \frac{1}{(2\pi h)^{d+1}} a\left(\frac{x+x'}{2},\xi\right) f^N(x',s)\chi(x-x')\exp\left\{\eta(G_p(x)-G_p(x'))\right\} \\ \exp\left\{\frac{i}{h}(\lambda G_p(x)-\lambda G_p(x')+\langle x-x',\xi\rangle)\right\}, \end{aligned} \tag{5.22}$$

où χ est une fonction lisse à support compact.

Le point stationnaire est $\{x=x',\ \xi=-\lambda d_x G\}$. La phase $\lambda(G_p(x)-G_p(x'))+\langle x-x',\xi\rangle$ est bien non-dégénérée en ce point. Le théorème [Hör03, 7.7.5] s'applique donc bien.

On commence par passer en revue les termes sous-principaux dans le développement asymptotique au point stationnaire. Étant donné que $dG_p \in \mathscr{C}^\infty(\widetilde{M})$, les contributions aux restes venant des dérivées de la phase seront $\mathcal{O}(h)$. De la même façons, les contributions des dérivées de a seront $\mathcal{O}(h)$. Celles de $\tilde{J}_p f_p^N$ seront $\mathcal{O}(h^k \tilde{J}(x)(1+G_p^+(x))^k)$.

Ensuite, après un nombre arbitraires d'intégrations par parties, il s'agit de montrer que loin du point stationnaire, on obtient un intégrand $\mathcal{O}(h^\infty)$ dans L^1. La première observation est que le support de l'intégrale en ξ est compact dans chaque fibre, et uniformément contrôlé. Après avoir coupé l'intégrale à distance h^ρ du point stationnaire, avec $0<\rho<1/2$, On va se ramener à vouloir estimer des intégrales de la forme

$$h^{k(1-2\rho)} \int \tilde{\chi}_h(x,x') e^{\eta(G(x)-G(x'))} \tilde{J}(x')(1+G_p^+)^k$$

où $\tilde{\chi}$ est une fonction lisse supportée dans le support de l'intégrand exprimé dans l'équation (5.22). Étant donné que $\tilde{J}$ est suffisemment régulière, ceci est encore $\mathcal{O}(h^{k(1-2\rho)}\tilde{J}(x)(1+G_p^+(x))^k)$. Dans ce reste, k peut être choisi arbitrairement grand, on prend $k=N/(1-2\rho)$.

Pour conclure, remarquons que si χ est supporté dans $\{|x|\leqslant C\}$, alors l'intégrale n'est supportée que si x est à distance $< C/2$ du support de σ.

□

Traitons le terme en δ_{ij} dans (5.19), qui correspond à $q=p$. Ce terme donne

$$\int_{\Gamma_p\backslash\widetilde{M}} e^{-2sG_p(x)} \tilde{J}_p^2(x) f_{\sigma,p}^N(x) f_p^N(x)dx \tag{5.23}$$

Sur la variété $\Gamma_p\backslash\widetilde{M}$ (qui est topologiquement un cylindre de base torique), on peut prendre les coordonnées $(t,x)\mapsto\varphi_p^t(x)$ avec x dans une horosphère projetée $\tilde{H}(p,t_0)$ avec $\{t_0\leqslant -\log b_i\}$. Le jacobien de cette transformation est exactement $\tilde{J}^2 e^{(t+t_0)d}$, ce qui fait que l'intégrale est égale à

$$\int_{\mathbb{R}} dt e^{(d-2s)(t+t_0)} \int_{\tilde{H}} \left\{f_{\sigma,p}^N . f_p^N\right\}\circ\varphi_t^p. \tag{5.24}$$

Étant donné les conditions sur le support de $f^N_{\sigma,p}$, et les bornes sur les f^N_p et $f^N_{\sigma,p}$, on déduit que cette intégrale est $\mathcal{O}(h^\infty)$.

Fixons désormais $[\gamma] \in \pi_1^{ij}(M)$. Alors le terme correspondant de la somme 5.19 s'écrit

$$\int_{\widetilde{M}} e^{-s(G_p+G_q\circ\gamma)} \tilde{J}_p f^N_{\sigma,p} \times \left\{ \tilde{J}_q f^N_q \right\} \circ \gamma.$$

D'après les propriétés d'équivariance (3.8) et (3.45), on peut réécrire ceci comme

$$\int_{\widetilde{M}} e^{-s(G_p+G_{\gamma^{-1}q})} \tilde{J}_p f^N_{\sigma,p} \times \tilde{J}_{\gamma^{-1}q} f^N_{\gamma^{-1}q}. \tag{5.25}$$

Avant de donner un développement asympotique pour cette intégrale, on considère $\{q\}$ une famille de points dans Λ^j_{par}, de sorte que les géodésiques reliant p à q quand q parcourt cette famille, forment une famille de représentant des géodésiques diffusées entre Z_i et Z_j. Ainsi, on récapitule les résultats obtenus jusqu'ici en :

$$\mathrm{Tr}^{ij}_\sigma(s) = \sum_{\{q\}} \int_{\widetilde{M}} e^{-s(G_p+G_q)} \tilde{J}_p f^N_{\sigma,p} \times \tilde{J}_q f^N_q + \mathcal{O}(h^N). \tag{5.26}$$

La preuve se termine avec celle du lemme suivant

Lemme 5.2.2. *Soit $p \in \Lambda^i_{par}$ et $q \in \Lambda^j_{par}$, et σ un symbole à support compact comme précédemment. Alors pour $N \geqslant 0$,*

$$\int_{\widetilde{M}} e^{-s(G_p+G_q)} \tilde{J}_p f^N_{\sigma,p} \times \tilde{J}_q f^N_q = (\pi h)^{d/2} \exp\left\{ -sT(\gamma) + \int_\gamma V_0 \right\} \left\{ \int_{-\lambda\gamma} \sigma + \mathcal{O}(h) \right\} + R_{p,q} \tag{5.27}$$

où γ est la géodésique diffusée représentée par p et q (dans cet ordre) ; $T(\gamma)$ est le temps de séjour de γ. Dans le membre de droite, l'intégrale est le long de la courbe $-\lambda\gamma$ vue comme une courbe dans T^*M. De plus, le reste $R_{p,q}$ vérifie

$$\sum_{\{q\}} R_{p,q} = \mathcal{O}(h^\infty). \tag{5.28}$$

Démonstration. Bien sûr, on obtient un développement en puissance de h, grâce à un argument de phase stationnaire ; d'après l'équation (5.21), le terme d'ordre h^n sera $\mathcal{O}(T(\gamma)^n)$, et le reste d'ordre h^N sera $\mathcal{O}(T(\gamma)^{2N})$. Néanmoins, seul le premier terme nous intéresse ici.

Tout d'abord, le long de la courbe γ (dans $\widetilde{M}$), $G_p + G_q$ est constante, de valeur $T(\gamma)$. On cherche donc à montrer que le hessien est non-dégénéré le long de la courbe

Nous allons utiliser les notations de la section 3.2.2 sur les champs de Jacobi le long de γ. Nous allons aussi utiliser des éléments de la preuve du lemme 3.3.4. D'après la relation (3.54), on trouve

$$\nabla^2(G_p + G_q) = \mathbb{U}_{x,\nabla G_p(x)} - \mathbb{S}_{x,\nabla G_p(x)}. \tag{5.29}$$

Ceci est non-dégénéré, donc l'intégrale est stationnaire et non-dégénérée le long de γ. Si on intègre seulement sur un voisinage de γ, le théorème de phase stationnaire s'applique, et on obtient que le premier terme dans l'asymptotique est l'intégrale le long de γ de

$$(2\pi h)^{d/2} \tilde{J}_p \tilde{J}_q \sigma(x, -\lambda\nabla_x G_p) \frac{1}{\sqrt{\det \nabla^2(G_p + G_q)}}. \tag{5.30}$$

Mais, d'après les équations (3.56) et (3.57), on trouve que

$$\frac{\tilde{J}_p \tilde{J}_q}{\sqrt{\det \nabla^2 (G_p + G_q)}} \tag{5.31}$$

est constant le long de γ. Il est donc égal à

$$2^{-d/2} \lim_{t \to +\infty} \tilde{J}_p(\varphi_t^p(x)) = 2^{-d/2} \exp \int_\gamma V_0. \tag{5.32}$$

Étant donné que nous disposons d'un contrôle raisonnable sur toutes les fonctions sous l'intégrale, nous savons déjà que l'intégrale a un développement comme annoncé au début de la preuve, modulo un reste qui représente la contribution de l'intégrale loin de la géodésique entre p et q. En faisant un nombre arbitraire d'intégrations par partie, étant donné que la norme du gradient de $(G_p + G_q)$ est uniformément minorée quand on se place à distance positive de la géodésique entre p et q, on obtient que ce reste $R_{p,q}$ est

$$\mathcal{O}(h^N) \int_{\widetilde{M}} e^{-\Re s(G_p+G_q)} \tilde{J}_p \tilde{J}_q (1 + G_p^+)^N (1 + G_q^+)^N \tag{5.33}$$

pour tout N. Mais si on somme ces intégrales sur $\{q\}$, en faisant le chemin inverse entre (5.7) et (5.19), on constate que l'on peut utiliser le lemme 3.1.5. En effet la somme pourra s'exprimer comme le produit scalaire entre P_{Z_i,V_0} et P_{Z_j,V_0}, qui converge étant donné que les deux sont dans L^2.

L'argument exposé au dessus fonctionne pour $N = 0$. Pour absorber les termes $(G_p^+)^N$, on peut utiliser la tactique déjà mise en œuvre dans la preuve du lemme 3.2.4. □

Il reste à sommer sur q. La somme correspondant aux termes principaux converge bien pour $\eta > \delta_g$. Étant donné que nous avons une borne $\mathcal{O}(h) \exp \int_\gamma V_0 - \eta T(\gamma)$ pour le reste, la somme des restes converge elle aussi pour $\eta > \delta_g$, et la preuve du lemme 5.1.1 est complète.

□

Appendix A

Annexes

A.1. Regularity of Horospheres for some Hadamard manifolds

Lemma A.1.1. *Let $\tilde{M}$ be a simply connected manifold of dimension $d+1$, with sectionnal curvature $-|K_{max}| < K < -|K_{min}| < 0$. Assume additionally that curvature tensor R of $\tilde{M}$ has all its covariant derivatives bounded. For a point $\xi \in S^*\tilde{M}$, we define $W^s(\xi)$ as $\{\xi' \in S^*M,\ d(\pi\varphi_t\xi', \pi\varphi_t\xi) \to 0$ as $t \to +\infty\}$, and similarly $W^u(\xi)$. Those are $\mathscr{C}^\infty$ submanifolds of $TS^*\tilde{M}$, uniformly in ξ; they form a continuous foliation of $TS^*\tilde{M}$, tangent respectively to E^s and E^u.*

Proof. We just check that the proof of the compact case also works for us.

Let $\xi \in S^*\tilde{M}$. Take $\nu > 0$, $t > 0$ and $0 < \epsilon < t$. For $k \geqslant 0$, let $\xi_k = \varphi_{kt}(\xi)$. Using the exponential charts for the Sasaki metric on $S^*\tilde{M}$, we can conjugate φ_t to diffeomorphisms from $T_{\xi_k}S^*\tilde{M}$ to $T_{\xi_{k+1}}S^*\tilde{M}$ that map 0 to 0. We still refer to those as φ_t. Let

$$\mathscr{H}_\nu := \{(z_k) \mid z_k \in T_{\xi_k}S^*\tilde{M},\ \limsup \|z_k\| e^{\nu k} < \infty\}.$$

This is a Banach space when endowed with

$$\|(z_k)\|_\nu := \sup \|z_k\| e^{\nu k}.$$

On $\mathscr{H}_\nu$, we can define

$$\Psi(z_k) := (0, \varphi_t(z_0) - z_1, \varphi_t(z_1) - z_2, \dots).$$

This is a $\mathscr{C}^\infty$ function on $\mathscr{H}_\nu$. We want to solve $\Psi = 0$ in $\mathscr{H}_\nu$. As the stable manifold ξ should be a graph over $E^s(\xi)$, for $(z_k) \in \mathscr{H}_\nu$, we decompose $(z_k) = (z_0^s, r)$. We need to show that $\partial_r \Psi$ is injective and surjective on a closed subspace, to use the implicit function theorem for Banach spaces. Let $V = (v_0^{u0}, v_1, \dots) \in \mathscr{H}_\nu$ where **u0** refers to the weak unstable direction $E^u \oplus \mathbb{R}\mathbf{X}$. We have

$$\partial_r \Psi(0) V = (0, d_\xi \varphi_t \cdot v_0^{u0} - v_1, d_{\xi_1} \varphi_t \cdot v_1 - v_2, \dots).$$

First, we prove this is injective. Assume $\partial_r \Psi(0) V = 0$. Then, we have $d_\xi \varphi_t \cdot v_0^{u0} = v_1$. Since the weak unstable direction is stable by the flow, $v_1 \in E^{u0}(\xi_1)$. By induction,

$v_k \in E^{u0}(\xi_k)$. However, V has to be in $\mathscr{H}_\nu$, so that there is a constant $C > 0$ such that for all $k > 0$,

$$\|v_0^{u0}\| = \|(d_\xi \varphi_{kt})^{-1} v_k\| \leqslant Ce^{-\nu k}.$$

This implies that $v_0^{u0} = 0$, and $V = 0$.

Now, we prove that $\partial_r \Psi$ is surjective on the space of sequences whose first term vanishes. Let $W = (w_0^{u0}, w_1, \dots) \in \mathscr{H}_\nu$. We decompose each $w_k = (w_k^s, w_k^{u0})$, and we try to solve $\partial_r \Psi V = W$, with $v_i = (v_i^s, v_i^{u0})$. If we find a solution, it has to satisfy for all $k > 0$,

$$v_k = d\varphi_{kt} v_0^{u0} - \sum_{l=0}^{k-1} d\varphi_{lt} w_{k-l}.$$

That is why we let

$$v_0^{u0} = \sum_{l=1}^{\infty} (d\varphi_{lt})^{-1} w_l^{u0}.$$

This sum converges because $\|(d\varphi_{kt})^{-1}|_{E^{u0}}\|$ is bounded independently of k, and we assumed w is in $\mathscr{H}_\nu$. Then, we have to check that the

$$v_k = -\sum_{l=1}^{k} d\varphi_{(k-l)t} w_l^s + \sum_{l=1}^{\infty} (d\varphi_{lt})^{-1} w_{k+l}^{u0}$$

define a sequence in $\mathscr{H}_\nu$. By the Anosov property, there are constants $\lambda > 0$ and $C > 0$ not depending on w, nor on ξ such that $\|d\varphi_{kt} w_l^s\| \leqslant e^{-\lambda kt} \|w_l^s\|$. So

$$\|v_k\| \leqslant \sum_{l=1}^{k} Ce^{-\lambda(k-l)t} e^{-l\nu} \|w\|_\nu + \sum_{l=1}^{\infty} e^{-(k+l)\nu} \|w\|_\nu.$$

It suffices to choose $\nu < \lambda t$, and we find that $v \in \mathscr{H}_\nu$.

By the Implicit Function Theorem, in a small enough neighbourhood of ξ, the strong stable manifold of ξ is a graph of a $\mathscr{C}^\infty$ function from $E^s(\xi) \to E^{u0}(\xi)$. Additionally, the derivatives of this function are controlled by the $\mathscr{C}^k$ norms of Ψ. These norms are the $\mathscr{C}^k$ norms of φ_t. They can be bounded independently of ξ — recall Ψ depends on ξ — according to Lemma B.1 and Proposition C.1 of [Bon14a]. Hence, the stable manifolds are uniformly smooth in the manifold.

Moreover, the tangent space of $W^s(\xi)$ at ξ has to be $E^s(\xi)$, according to the dynamical definition of $W^s(\xi)$. We deduce that the regularity of the lamination formed by the collection of $W^s(\xi)$, $\xi \in S^*M$ has the regularity of the *splitting* $E^s \oplus E^u$. Using the description as Green's fiber bundles for E^s and E^u, one can prove that they are Hölder, and the lamination is actually a foliation.

The case of unstable manifolds is similar. □

A.2. Functional spaces in a cusp

First, let us recall some definitions on covariant derivatives. If S is a tensor on a riemannian manifold, one defines its covariant derivative in the following way:

$$(\nabla_X S)(Y_1, \dots, Y_n) := X\left(S(Y_1, \dots, Y_n)\right) - \sum_i S(Y_1, \dots, \nabla_X Y_i, \dots, Y_n)$$

In particular, when f is a function on a riemannian manifold N, one defines a family of tensors $\nabla^n f$ in the following way.

$$\nabla f : X \mapsto X(f) \quad \text{and} \quad \nabla^{n+1}_{X_0,\dots,X_n} f := (\nabla_{X_0}\nabla^n)_{X_1,\dots X_n} f.$$

We also define it for vectors — which are $(0,1)$ tensors:

$$\nabla Z : X \mapsto \nabla_X Z \quad \text{and} \quad \nabla^{n+1}_{X_0,\dots,X_n} Z := (\nabla_{X_0}\nabla^n)_{X_1,\dots X_n} Z.$$

This enables us to define, for $x \in N$

$$\|\nabla^n f\|(x) = \sup_{X_1,\dots,X_n \in T_x N} \frac{|\nabla^n f(X_1,\dots,X_n)|}{\|X_1\|\dots\|X_n\|}$$

Then, the space $\mathscr{C}^n(N)$ is the set of functions on N that are C^n, and such that

$$\|f\|_{\mathscr{C}^n(N)} := \sum_{k=0}^{n} \sup_{x\in N} \|\nabla^k f\|(x) < \infty.$$

Now, we turn to Sobolev spaces. When N is complete, $L^2(N)$ is a Hilbert space. For $n \geqslant 0$ an integer, one defines the norm

$$\|f\|^2_{H^n(Z)} := \sum_{k\leqslant n} \left\|\|\nabla^k f\|(x)\right\|^2_{L^2(dx)}$$

The Sobolev space $H^n(N)$ of order n is the completion of $C^\infty(N)$ for this norm. If N has no boundary, then $H^{-n}(N)$ is defined as the dual of $H^n(N)$.

Using the Lax-Milgram theorem, exactly as for the Laplacian on $\mathbb{R}^n$, one proves that for any $\epsilon > 0$, $-\Delta + \epsilon$ is invertible on $H^1(N)$ with values in $H^{-1}(N)$. Since it is also positive, one can use the spectral theorem to define $(-\Delta+1)^s$ for any $s \in \mathbb{R}$. One observes that $\|.\|_{H^1(N)}$ and $\|(-\Delta+1)^{1/2}.\|$ are equivalent norms on $H^1(N)$.

The cusp Z is complete, so the above apply. Now, one can compute the following :

$$\begin{aligned} \nabla_{X_y} X_y &= 0 \\ \nabla_{X_y} X_{\theta_i} &= 0 \\ \nabla_{X_{\theta_i}} X_y &= -X_{\theta_i} \\ \nabla_{X_{\theta_i}} X_{\theta_j} &= \delta_{ij} X_y \end{aligned} \tag{A.1}$$

From this and the definition of ∇^n, if α is a space-index of length n, we find

$$\nabla^n_{X_{\alpha_1},\dots,X_{\alpha_n}} f = X_\alpha f + \sum_\beta \pm X_\beta f$$

where β are other space-indices, of length $< n$. Whence by induction on $n \geqslant 0$ we find

$$\|f\|_{\mathscr{C}^n(Z)} \text{ is equivalent to } \sum_{|\alpha|\leqslant n} \|X_\alpha f\|_{L^\infty(Z)}. \tag{A.2}$$

and

$$\|f\|_{H^n(Z)} \text{ is equivalent to } \sum_{|\alpha|\leqslant n} \|X_\alpha f\|_{L^2(Z)}. \tag{A.3}$$

Now, we define, for s a real number

$$\|f\|_s := \|(-\Delta + 1)^s f\|_{L^2}$$

We want to show that the completion of $C^\infty(Z)$ *is* the sobolev space $H^s(Z)$ for integer s, and that then, $\|.\|_s$ is equivalent to $\|.\|_{H^s(Z)}$. Such a result is deduced of an elliptic estimate similar to that in pp 358 in [Tay11]. Actually, the proof therein adapts to a cusp if one defines the slope operators $D_{j,h}$ in the following way

$$D_{j,h}f(x) = \frac{1}{h}\left(f(x + hX_j) - f(x)\right), \quad j = y, \theta_1, \ldots, \theta_d.$$

Then, using $P = -h^2\Delta/2$, we also define the semi-classical Sobolev norms :

$$\|f\|_{s,h} := \|(P + 1)^{s/2} f\|_{L^2(Z)}.$$

One gets for some constant $C > 0$

$$\frac{1}{C} h^{s^+} \|f\|_s \leqslant \|f\|_{s,h} \leqslant C h^{-s^-} \|f\|_s. \tag{A.4}$$

where s^+ and s^- are the positive and negative part of s.

To finish this section, we define the non-integer Sobolev spaces using complex interpolation — as in pp 321 from [Tay11].

A.3. On the Sasaki metric

A.3.1. The curvature tensor of a Sasaki metric

There is a useful — and easily accessible — reference for the Sasaki metric on tangent spaces: [GK02]. We are going to rely heavily on it to avoid introducing too much machinery — we also the notations from there, which are not always consistent with ours in the rest of this article. In the following paragraph, we retain the notations therein. We want to show that proposition A.4.1 applies to the geodesic flow of cusp manifolds. We prove

Proposition A.3.1. *Assume that the curvature tensor of M is bounded, and all its covariant derivatives also. For $R > 0$, let $TM_R := \{v \in TM \mid \|v\|^2 \leqslant 2R\}$ be endowed with the Sasaki metric. Then*

1. *The curvature tensor of TM_R, and all its derivatives are bounded.*

2. *It is also the case for the vector of the geodesic flow*

Remark that when the curvature of M is constant, the covariant derivative of the curvature tensor is just 0, so the above proposition applies to cusp manifolds — and more generally to any geometrically finite manifold with hyperbolic ends.

Proof. We denote (p, u) for points of T^*M. If X is a vector in T_pM, we denote by X^h (resp. X^v) its horizontal (resp. vertical) lift, which are vectors in $T_{(p,u)}TM$.

Let T be a vector valued tensor on M, $T \in \Gamma(M, TM \otimes T(T^*M))$. From T we can construct a variety of vector valued on TM. Indeed, first, we can construct tensors on M valued in TTM by taking either the vertical of the horizontal lift of T. Then, we can compose T by either $X^v \mapsto X$ or $X^h \mapsto X$ and we obtain an element of $\Gamma(TM, TTM \otimes T(T^*TM))$. We consider now the class $\mathscr{B}_0$ of tensors on TM that are obtained in this way when T and all its derivatives are bounded. We also require that 0-tensors u^h and u^v are in $\mathscr{B}_0$. Now, $\mathscr{B}$ is the smallest class of tensors stable by composition and sums that contains $\mathscr{B}_0$.

From the formulae page 16 (prop. 7.5) for the curvature tensor of the Sasaki metric, we see that it is in $\mathscr{B}$ since the curvature tensor R of M as well as all its derivatives are bounded. The vector of the geodesic flow also is in $\mathscr{B}$ because it is $V(p, u) = u^h$.

We want to prove that $\mathscr{B}$ is *stable under covariant derivatives*, as it suffices to end the proof. We work in local coordinates. Observe that since covariant derivatives behave well with composition and sums, it suffices to prove that covariant derivatives of elements of $\mathscr{B}_0$ are in $\mathscr{B}$.

Let $p \in M$, let U be some small open set containing p where the normal coordinates at p, $\exp_p^{-1} : U \to T_pM$ are well defined. Taking an orthonormal basis $X_1, \dots, X_n$ in T_pM, we have coordinates $x_1, \dots, x_n$ on U. Then, we can consider coordinates $v_1, \dots, v_{2n}$ on TU as in page 6 of [GK02]. Since we have taken normal coordinates, the Christoffel coefficients vanish at p, and we have (see lemma 4.3 p. 7)

$$\partial_{x_i}(p)^h = \partial_{v_i} \quad \partial_{x_i}(p)^v = \partial_{v_{n+i}}.$$

At a point (p', u), we have

$$u = \sum v_{n+k} \partial_{x_k}(p'). \tag{A.5}$$

Since we have taken normal coordinates, the $\nabla_{\partial_{x_1}} \partial_{x_i}$ vanish at p. From this and the formulae for covariant derivatives in proposition 7.2 page 15, we find

$$\nabla_{a^h + b^v} u^h = b^h + \frac{1}{2}(R_p(u, b)u)^h - \frac{1}{2}(R_p(a, u)u)^v.$$

and

$$\nabla_{a^h + b^v} u^v = b^v + \frac{1}{2}(R_p(u, u)a)^h.$$

Now, we take T a tensor on M with all its derivatives bounded, and we just consider the case when T is a 1 tensor, and $T'(a^h + b^v) = (T(a))^h$. This defines an element of $\mathscr{B}_0$.

$$(\nabla_{X^h + Y^v} T')(a^h + b^v) = \nabla_{X^h + Y^v} (T(a))^h - T'(\nabla_{X^h + Y^v}(a^h + b^v)).$$

Using again the formulae for Sasaki covariant derivatives, we can expand this expression. There will be terms containing $\nabla_X T$ and terms involving R_p, u and T, so the result *will* be an element of $\mathscr{B}$.

To give a complete proof, we would have to consider all the possibilities that lead to similar computations; we leave this as an exercise for the reader. □

A.3.2. The Sasaki metric in a cusp, and symbols

Now, M is a cusp manifold. The Sasaki metric is *a priori* defined on the tangent space. However, there is a correspondance $v \mapsto \langle v, . \rangle$ between TM and T^*M, and we define the Sasaki metric on T^*M by pushing forward the metric on TM. As a consequence, T^*M is endowed with a connection ∇ and $\mathscr{C}^k$ norms. The following fact is the key to proving the Egorov lemma 2.3.

Proposition A.3.2. *Take $E > 0$, and consider functions on T^*M supported in $(T^*M)_E := \{p(\xi) \leqslant E\}$. For such functions, the $\mathscr{C}^k(T^*M)$ norm is equivalent to the norm given by symbol estimates with $k' \leqslant k$ derivatives.*

Proof. The part of $(T^*M)_E$ above the compact part of M is relatively compact, so all C^k norms over it are equivalent. We just have to work in the cusps. Let us first start by finding the expression for the Sasaki metric in a cusp Z; we use again [GK02]. We have coordinates y, θ, and the coordinates on the tangent space v_y, v_θ. From (A.1), we can compute

$$\begin{aligned} y\nabla_{\partial_y}\partial_y &= -\partial_y \\ y\nabla_{\partial_y}\partial_{\theta_i} &= -\partial_{\theta_i} \\ y\nabla_{\partial_{\theta_i}}\partial_y &= -\partial_{\theta_i} \\ y\nabla_{\partial_{\theta_i}}\partial_{\theta_j} &= \delta_{ij}\partial_y \end{aligned}$$

We deduce that the Sasaki metric on TM is

$$g = \frac{1}{y^2}\left(dy^2 + d\theta^2 + (dv_y + \frac{1}{y}(v_\theta.d\theta - v_y dy))^2 + (dv_\theta - \frac{1}{y}(v_\theta dy + v_y d\theta))^2\right)$$

Now, $v_y = y^2 Y$, and $v_\theta = y^2 J$, so this gives on T^*M

$$g = \frac{dy^2 + d\theta^2}{y^2} + y^2\left((dY + \frac{1}{y}(J.d\theta - Ydy))^2 + (dJ - \frac{1}{y}(Jdy + Yd\theta))^2\right)$$

Recall that $p = |\xi|^2/2$ is the symbol of $-h^2\Delta/2$, and $X_y = y\partial_y$, $X_\theta = y\partial_\theta$, $yX_Y = \partial_Y$, $yX_J = \partial_J$. We get that

$$g(X_y) = g(X_{\theta_i}) = 1 + 2p \quad \text{and} \quad g(X_Y) = g(X_{J_i}) = 1.$$

and when $k \neq i$ — $\langle ., . \rangle$ being the scalar product,

$$\langle X_y, X_{\theta_i} \rangle = \langle X_{\theta_k}, X_{\theta_i} \rangle = \langle X_Y, X_{J_i} \rangle = \langle X_{J_k}, X_{J_i} \rangle = 0$$

$$\langle X_y, X_Y \rangle = -yY, \quad \langle X_y, X_{J_i} \rangle = -yJ_i, \quad \langle X_{\theta_i}, X_Y \rangle = yJ_i, \quad \langle X_{\theta_i}, X_{J_k} \rangle = -\delta_{ik} yY.$$

If we use the Koszul formula [Pau14] to determine the covariant derivatives of $X_{y,\theta,Y,J}$, we will find that they are of the type $aX_y + bX_\theta + cX_Y + dX_J$, where a, b, c, d are elements of $S^1_{\mathcal{V}}$ — defined in the paragraph after (2.5). As a consequence, if α is a finite sequence of $\alpha_j \in \{y, \theta_i, Y, J_j\}$ of length k, there are symbols $f_\beta \in S_{\mathcal{V}}$ for all sequences β of the same type, of length $k' < k$, such that f_β is of order $\leqslant k - k'$, and

$$\nabla^k_{X_{\alpha_1} \ldots X_{\alpha_k}} = X_\alpha + \sum_\beta f_\beta X_\beta$$

From this we deduce that on $(T^*M)_E$, the norms

$$\Big\{ \sum_{|\alpha|\leqslant k} q_{n,\alpha} \Big\}_n \text{ and } \sum_{|\alpha|\leqslant k} \sup_{T^*M} \|\nabla^k_{X_\alpha}\|$$

are equivalent. We are left to prove that the latter is equivalent to the $\mathscr{C}^k(T^*M)$ norm. It is *a priori* bounded by it, so we need to prove a lower bound.

We have coordinates in each $T_\xi T^*Z$ given by

$$T_\xi T^*Z \ni X = u_y X_y + u_\theta X_\theta + u_Y X_Y + u_J X_J$$

this defines a map $u_\xi : T_\xi T^*Z \to \mathbb{R}^{2d+2}$. Let us endow $\mathbb{R}^{2d+2}$ with the Euclidean metric. The equivalence to the $\mathscr{C}^k(T^*M)$ norm is assured if both u_ξ and u_ξ^{-1} are bounded independently of ξ as long as $p(\xi) \leqslant E$. Let us compute:

$$\|u_y X_y + u_\theta X_\theta + u_Y X_Y + u_J X_J\|^2 = u_y^2 + u_\theta^2 + (u_Y + yJ.u_\theta - yYu_y)^2 + (u_J - yJu_y - yYu_\theta)^2.$$

This is a bounded, positive quadratic form q on $\mathbb{R}^{2d+2}$. To end the proof, we need to show that there is some $C > 0$ such that $q > C.Id$ independently of ξ, when $p(\xi) \leqslant E$. However, since q is bounded by $1 + 2p$, it suffices to prove that its determinant is bigger than some positive constant not depending on ξ as long as $p(\xi) \leqslant E$. Some elementary computations show that the determinant is actually

$$(1 + y^2J^2)^{d-1} \geqslant 1.$$

□

A.4. Estimating the derivatives of a flow on a Riemannian manifold

The following proposition should be classical, but for lack of a reference, we enclose a proof.

Proposition A.4.1. *Let φ^t be a flow in a manifold N, such that all the covariant derivatives of the vector field V of the flow and of the curvature tensor of N are bounded. Assume also that the maximal Lyapunov exponent λ_0 of φ^t — as defined in 2.2.5 — is* finite. *Then for all $\lambda > \lambda_0$, there are constants $C_n > 0$, such that for $f \in \mathscr{C}^n(N)$, for $t \in \mathbb{R}$,*

$$\|f \circ \varphi_t\|_{\mathscr{C}^n(N)} \leqslant C_n e^{n\lambda|t|} \|f\|_{\mathscr{C}^n(N)}$$

The proof is inspired by [DG14], which itself comes from [BR02] ($N = \mathbb{R}^n$). In the usual proofs of this type of result, at some point, one uses coordinates to transport the problem to $\mathbb{R}^n$. When N is compact, this is reasonnable because all metrics on N are equivalent. When N is non compact, it is probably possible to take a similar approach. However, one would have be careful and take coordinate charts with derivatives nicely bounded. We chose to avoid taking coordinates altogether, and give an intrinsic formulation of the proof, hence the appearance of many tensors.

The main idea of the proof is to avoid estimating higher derivatives of the flow, and replace them by higher derivatives of the vector field of the flow.

Proof. We want to compute

$$\nabla_{X_1,\dots X_n}(f\circ\varphi^t).$$

We are going to compare this with

$$\nabla_{\varphi^t_* X_1,\dots \varphi^t_* X_n} f.$$

In the first expression, there are *a priori*, higher derivatives of the flow, while the second one only contains first order derivatives that are much easier to estimate. Recall that $\varphi^t_* X$ is the pullback of X by φ^{-t}, i.e $(d\varphi^{-t})^{-1}.(X\circ\varphi^{-t})$. Let $z\in N$, and $X_1,\dots,X_n\in T_zN$. Let

$$W^n_t(X_1,\dots X_n)f := \left[(\varphi^t)^*\left(\nabla_{\varphi^t_* X_1,\dots,\varphi^t_* X_n}\right)\right]f$$

By definition, this means

$$[W^n_t(X_1,\dots,X_n)f]\circ\varphi^{-t} = \left(\nabla_{\varphi^t_* X_1,\dots,\varphi^t_* X_n}\right)(f\circ\varphi^{-t}).$$

From the definition, we see that $W^n_t f$ is a tensor. We observe that

$$W^n_t = \nabla W^{n-1}_t + \sum_{i=2,\dots n} W^{n-1}_t(X_2,\dots,\nabla_{X_1}X_i - (\varphi^t)^*(\nabla_{\varphi^t_* X_1}\varphi^t_* X_i),\dots,X_n). \tag{A.6}$$

One can compute $(\varphi^t)^*(\nabla_{\varphi^t_* X_1}\varphi^t_* X_i)$. Indeed, consider the fact

$$\partial_t(\varphi^t)^*X(t) = (\varphi^t)^*[V,X(t)] + (\varphi^t)^*\partial_t X$$

We deduce that

$$\partial_t(\varphi^t)^*(\nabla_{\varphi^t_* X}\varphi^t_* Y) = (\varphi^t)^*\left([V,\nabla_{\varphi^t_* X}\varphi^t_* Y] - \nabla_{[V,\varphi^t_* X]}\varphi^t_* Y - \nabla_{\varphi^t_* X}[V,\varphi^t_* Y]\right).$$

That is

$$\partial_t(\varphi^t)^*(\nabla_{\varphi^t_* X}\varphi^t_* Y) = (\varphi^t)^* Z(\varphi^t_* X,\varphi^t_* Y)$$

with

$$Z(\varphi^t_* X,\varphi^t_* Y) = \nabla^2_{\varphi^t_* X \varphi^t_* Y} V + R_\nabla(V,\varphi^t_* X)\varphi^t_* Y,$$

where R_∇ is the curvature tensor. So,

$$\nabla_X Y - (\varphi^t)^*(\nabla_{\varphi^t_* X}\varphi^t_* Y) = -\int_0^t (\varphi^s)^* Z(\varphi^s_* X,\varphi^s_* Y)ds \tag{A.7}$$

This equation we found in [DG14], and the rest of the proof is devoted to proving similar formulae for higher order derivatives. Let us call the tensor in the RHS $L_t(X,Y)$. We can already compute explicitly

$$W^1_t f = X_1 f \quad W^2_t f = \nabla_{X_1,X_2} f + L_t(X_1,X_2)f.$$

If $F\to N$ is a vector bundle over N, we write $T(F)$ for the tensor algebra of F. Now, we introduce a class of tensors $\mathscr{T}$. Elements of $\mathscr{T}$ take the following form: $T: s,t\mapsto T(s,t)\in\Gamma(N,TN\otimes T(T^*N))$. First, the identity is in $\mathscr{T}$. Second, if $T_1(s,t),\dots,T_k(s,t)$

are in $\mathscr{T}$ with $k \geqslant 2$, and if $R \in \Gamma(N, TN \otimes (T^*N)^{\otimes k})$ with all its covariant derivatives bounded, then

$$T^R_{T_1,\dots,T_k}(s,t) := \int_s^t (\varphi^u)^* R(\varphi^u_* T_1(u,t), \dots, \varphi^u_* T_k(u,t)) du \tag{A.8}$$

is also in $\mathscr{T}$. Finally, we require that $\mathscr{T}$ is the smallest vector space containing all the tensors described above. For example, L_t is $T^{-Z}_{Id,Id}(0,t)$; we denote $L(s,t) = T^{-Z}_{Id,Id}(s,t)$. Then

Lemma A.4.2. *$W^n_t f$ can be written as a sum of terms*

$$\nabla^k f(T_1(0,t), \dots, T_k(0,t)) \tag{A.9}$$

where the T_i's are in $\mathscr{T}$ of the correct order (that is, T_i takes values in $\Gamma(N, TN \otimes (T^*N)^{\otimes n_i}$ with $n_1 + \dots + n_k = n$).

Proof. We have already checked it for $n = 1$ and $n = 2$. Actually, we will check that if A_t is an $(n-1)$-tensorial operator of the type (A.9), then

$$\nabla A_t - \sum_{i=1,\dots,n} A_t(X_2, \dots, L_t(X_1, X_i), \dots, X_n)$$

is a sum of such operators (of orders n and $n-1$). Let us observe that

$$\begin{aligned}\nabla\left[\nabla^k(T_1(0,t), \dots, T_k(0,t))\right] &= \nabla^{k+1}(Id, T_1(0,t), \dots, T_k(0,t)) \\ &\quad - \sum_{i=1,\dots,k} \nabla^k(T_1(0,t), \dots, \nabla T_i(0,t), \dots, T_k(0,t)).\end{aligned}$$

We deduce that it suffices to show that when $T \in \mathscr{T}$ is a k-tensor,

$$\begin{aligned}T' := \nabla_X T(s,t) &+ \int_0^s (\varphi^w)^* Z(\varphi^w_* X, \varphi^w_* T(s,t)) dw \\ &+ \sum_{i=1}^k T(0,t)(X_1, \dots, L_t(X_0, X_i), \dots, X_k)\end{aligned}$$

is in $\mathscr{T}$. We prove this by induction on k. First, if $k = 1$, T is the identity, and we find $T'(s,t) = L(s,t)$.

Assume we are done for all $k \leqslant n$. Then, let T be a $n+1$ tensor in $\mathscr{T}$. By construction, it is a sum of terms as in (A.8). Since the property we are trying to prove is stable by taking sums, assume there is only one term in the sum. The T_i's all are of order $< n+1$, and we can compute, using (A.7) in the first line

$$\begin{aligned}\nabla_X T(s,t) &= \int_s^t \int_0^u (\varphi^{u-w})^*(-Z)(\varphi_*^{u-w}X, \varphi_*^{-w}R(\varphi_*^u T_1(u,t),\dots,\varphi_*^u T_k(u,t)))dwdu \\ &+ \int_s^t (\varphi^u)^* \nabla_{\varphi_*^u X}\big(R(\varphi_*^u T_1(u,t),\dots,\varphi_*^u T_k(u,t))\big)du. \\ \nabla_X T(s,t) &= T_{Id,T}^{-Z}(s,t) + \int_0^s (\varphi^w)^*(-Z)(\varphi_*^w X, \varphi_*^w T(s,t))dw \\ &+ \int_s^t (\varphi^u)^*(\nabla_{\varphi_*^u X} R)(\varphi_*^u T_k(u,t),\dots,\varphi_*^u T_k(u,t))du. \\ &+ \sum_{i=1}^k \int_s^t (\varphi^u)^* R(\varphi_*^u T_1(u,t),\dots,\nabla_{\varphi_*^u X}\varphi_*^u T_i(u,t),\dots,\varphi_*^u T_k(u,t))du.\end{aligned}$$

Hence we find

$$\begin{aligned}\nabla_X T(s,t) + \int_0^s (\varphi^w)^* Z(\varphi_*^w X, \varphi_*^w T(s,t))dw &= T_{Id,T}^{-Z}(s,t) + T_{Id,T_1,\dots,T_k}^{\nabla R}(s,t) \\ &+ \sum_{i=1}^k \int_s^t (\varphi^u)^* R(\varphi_*^u T_1(u,t),\dots,\nabla_{\varphi_*^u X}\varphi_*^u T_i(u,t),\dots,\varphi_*^u T_k(u,t))du\end{aligned}$$

But we precisely have

$$(\varphi^u)^* \nabla_{\varphi_*^u X}\varphi_*^u T_i(u,t) = \nabla_X T_i(u,t) + \int_0^u (\varphi^w)^* Z(\varphi_*^w X, \varphi_*^w T_i(u,t))dw.$$

so we can use the induction hypothesis, and conclude.

□

Lemma A.4.3. *When $T \in \mathscr{T}$ is a n-tensor, there is a constant $C > 0$ such that whenever $0 \leqslant s \leqslant t$,*

$$\|\varphi_*^s T(s,t)((\varphi^t)^* X_1,\dots,(\varphi^t)^* X_n)\| \leqslant C e^{n\lambda(t-s)}\|X_1\|\dots\|X_n\|.$$

Proof. We proceed by induction. First, for the identity, this is true because the maximal lyapunov exponent of the flow is bounded. Now, we assume it is true for all k-tensors in $\mathscr{T}$ with $k \leqslant n$, and let $T \in \mathscr{T}$ be a $n+1$ tensor.

$$\|\varphi_*^s T(s,t)\| \leqslant \int_s^t e^{\lambda(u-s)} \prod_{i=1}^k \|\varphi_*^u T_i(u,t)\|du$$

If we use the induction hypothesis, we get

$$\|\varphi_*^s T(s,t)\| \leqslant C\|X_1\|\dots\|X_n\| \underbrace{\int_s^t e^{\lambda(u-s)+(n+1)\lambda(t-u)}du}_{\leqslant Ce^{\lambda(n+1)(t-s)}}$$

□

We conclude the proof by observing that

$$\nabla^n(f\circ\varphi_{-t})(X_1,\dots,X_n) = W_t^n f((\varphi^t)^* X_1,\dots,(\varphi^t)^* X_n)$$

□

A.5. Estimating the regularity of solutions for transport equations

Lemma A.5.1. *Let N be a riemannian manifold such that all the derivatives of its curvature are bounded. Let G be a C^∞ function on N, such that $\|\nabla G\| = 1$, and $\|\nabla G\|_{\mathscr{C}^n(N)}$ is bounded for all n. Let φ_t^G be the flow generated by $V = \nabla G$. Assume that φ^G is expanding, that is, there is a $\lambda > 0$ such that if $\mathbf{u} \perp V$, $\|d\varphi_t^G.u\| \geqslant Ce^{\lambda t}\|u\|$ for $t > 0$.*

Let g_0 be a C^∞ function on N, supported in $G \geqslant \ell$. Let

$$g_1(x) = \int_{-\infty}^0 g_0 \circ \varphi_t^G dt.$$

Then if $\mathscr{L}(\tau) = \sup\{|g_0(x)|, G(x) = \tau\}$, for all n there is a constant $C_n > 0$ only depending on G such that

$$|g_1(x)| \leqslant \int_\ell^{G(x)} \mathscr{L}(\tau)d\tau, \quad \|\nabla g_1\|_{\mathscr{C}^{n-1}(G\leqslant t)} \leqslant C_n \|g_0\|_{\mathscr{C}^n(G\leqslant t)}$$

Proof. The first part of the statement is obvious. We concentrate on the second part. The basic idea is that when differentiating in the direction of the flow, one obtains g_0, and when differentiating in other directions, one can use the contracting properties of φ_t^G in negative time. Let $x \in N$, and $X_1, \dots, X_n$ vectors at x. We want to evaluate $\nabla_{X_1,\dots,X_n} g_1(x)$. We can decompose the X_i's according to

$$T_x N = \mathbb{R}V \oplus V^\perp.$$

By linearity, we can assume that either $X_i \propto V$ or $X_i \perp V$. Additionally, we assume $\|X_i\| = 1$. By taking symmetric parts, and antisymmetric parts of ∇, we see that it suffices to evaluate $\nabla_{X_1,\dots,X_n} g_1$ when the X_i's colinear to V are the last in the list. That corresponds to differentiating g_1 first along V. Now, there are two cases. First, one the X_i's is colinear to V. Then

$$\nabla_{X_1,\dots,V} g_1 = \nabla_{X_1,\dots,X_{n-1}} g_0$$

We are left to consider the case when all the X_i's are orthogonal to V. For this, we use the proof from [Bon14a, appendix B]. From therein, we know that for $t > 0$,

$$\nabla_{X_1,\dots,X_n}(g_0 \circ \varphi_{-t}^G) = W_t^n g_0((\varphi_t^G)^* X_1, \dots, (\varphi_t^G)^* X_n)$$

Where — lemma B.2 — $W_t^n g_0$ is a sum of tensors of the form

$$\nabla^k_{T_1(0,t),\dots,T_k(0,t)} g_0.$$

The $T_i(s,t)$'s are tensors with a particular structure. Either it is of order 1 and $T_i(s,t)(X) = X$, or it is of higher order and

$$T_i(s,t) = \int_s^t (\varphi_u^G)^* R_i \left[(\varphi_u^G)_* T_{i,1}(u,t), \dots, (\varphi_u^G)_* T_{i,k_i}(u,t)\right] du$$

where R_i is a tensor bounded with all derivatives bounded, and the $T_{i,j}$ have the same structure. Observe that for $X \in TN$, $\|(\varphi_t^G)^* X\| \leqslant C\|X\|$ when $t > 0$, and $\|(\varphi_t^G)^* X\| \leqslant Ce^{-\lambda t}\|X\|$ when $X \perp V$. By induction, we deduce that for $t > 0$,

$$\|\nabla_{X_1,\dots,X_n}(g_0 \circ \varphi_{-t}^G)\|(x) \leqslant C_n e^{-\lambda n t}\|g_0\|_{\mathscr{C}^n(G \leqslant G(x))}.$$

We just have to integrate this for $t \in [0, +\infty[$, and the exponential decay ensures the convergence. □

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